Statistical Mechanics of Lattice Systems

A Concrete Mathematical Introduction

Sacha Friedli and Yvan Velenik

"Rigorous statistical mechanics has a long tradition, starting in the 1960s with Dobrushin, Lanford, Ruelle, Sinai and many other mathematical physicists. Yet, surprisingly, there has not been an up-to-date textbook on an introductory level. This gap is bridged masterly by the present book. It addresses the curious newcomer by employing a carefully designed structure, which starts from a physics introduction and then uses the two-dimensional Ising model as a stepping-stone towards the richness of the subject. The book will be enjoyed by students and researchers with an interest in either theoretical condensed matter physics or probability theory."
Herbert Spohn, Technische Universität München

"An excellent introduction to classical equilibrium statistical mechanics. There is clear concern with the understanding of ideas and concepts behind the various tools and models. Topics are presented through examples, making this book a genuine 'concrete mathematical introduction.' While reading it, my first wish was to use it as soon as possible in a course. It is a 'must' for the libraries of graduate programs in mathematics or mathematical physics."
Maria Eulália Vares, Universidade Federal do Rio de Janeiro

"This book is a marvelous introduction to equilibrium statistical mechanics for mathematically inclined readers, which does not sacrifice clarity in the pursuit of mathematical rigor. The book starts with basic definitions and a crash course in thermodynamics, and gets to sophisticated topics such as cluster expansions, the Pirogov–Sinai theory and infinite volume Gibbs measures through the discussion of concrete models. This book should be on the bookshelf of any serious student, researcher and teacher of mathematical statistical mechanics."
Ofer Zeitouni, Weizmann Institute

"Sacha Friedli and Yvan Velenik have succeeded in writing a unique, modern treatise on equilibrium statistical mechanics. They cover many fundamental concepts, techniques and examples in a didactic manner, providing a remarkable source of knowledge, grounded and refined by years of experience. Stimulating and appealing, this book is likely to inspire generations of students and scientists."
Francis Comets, Université Paris Diderot

Cover designed by Rob Lock after a proposal by Z+Z

CAMBRIDGE
UNIVERSITY PRESS
www.cambridge.org

ISBN 978-1-107-18482-4

Friedli and Velenik

Statistical Mechanics of Lattice Systems

CAMBRIDGE

Statistical Mechanics of Lattice Systems

A Concrete Mathematical Introduction

S. FRIEDLI
Universidade Federal de Minas Gerais, Brazil

Y. VELENIK
Université de Genève

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India

79 Anson Road, #06-04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107184824
DOI: 10.1017/9781316882603

First published 2018

Printed in the United States of America by Sheridan Books, Inc.

A catalog record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data
Names: Friedli, Sacha, 1974– author. | Velenik, Yvan, 1970– author.
Title: Statistical mechanics of lattice systems : a concrete mathematical introduction / Sacha Friedli (Universidade Federal de Minas Gerais, Brazil), Yvan Velenik (Université de Genáeve).
Description: Cambridge, United Kingdom ; New York, NY : Cambridge University Press, 2017.
Identifiers: LCCN 2017031217| ISBN 9781107184824 (hardback) | ISBN 1107184827 (hardback)
Subjects: LCSH: Statistical mechanics.
Classification: LCC QC174.8 .F65 2017 | DDC 530.13–dc23
LC record available at https://lccn.loc.gov/2017031217

ISBN 978-1-107-18482-4 Hardback

Statistical Mechanics of Lattice Systems

A Concrete Mathematical Introduction

This motivating textbook gives a friendly, rigorous introduction to fundamental concepts in equilibrium statistical mechanics, covering a selection of specific models, including the Curie–Weiss and Ising models, the Gaussian Free Field, $O(N)$ models, and models with Kać interactions. Using classical concepts such as Gibbs measures, pressure, free energy and entropy, the book describes the main features of the classical description of large systems in equilibrium, in particular the central problem of phase transitions. It treats such important topics as Peierls' argument, the Dobrushin Uniqueness, Mermin–Wagner and Lee–Yang theorems, and develops from scratch such workhorses as correlation inequalities, the cluster expansion, Pirogov–Sinai theory and reflection positivity. Written as a self-contained course for advanced undergraduate or beginning graduate students, the detailed explanations, large collection of exercises (with solutions), and appendix of mathematical results and concepts also make it a handy reference for researchers in related areas.

Sacha Friedli is Associate Professor of Mathematics at the Federal University of Minas Gerais in Brazil. His current research interests are in statistical mechanics, mathematical physics and Markov processes.

Yvan Velenik is Professor of Mathematics at the University of Geneva. His current work focuses on applications of probability theory to the study of classical statistical mechanics, especially lattice random fields and random walks.

For the sunshine and smiles: Janet, Jean-Pierre, Mimi and Kathryn.

Aos amigos e colegas do Departamento de Matemática.

À Agnese, Laure et Alexandre, ainsi qu'à mes parents.

Preface

Equilibrium statistical mechanics is a field that has existed for more than a century. Its origins lie in the search for a microscopic justification of equilibrium thermodynamics, and it developed into a well-established branch of mathematics in the second half of the twentieth century. The ideas and methods that it introduced to treat systems with many components have now permeated many areas of science and engineering, and have had an important impact on several branches of mathematics.

There exist many good introductions to this theory designed for physics undergraduates. It might, however, come as a surprise that textbooks addressing it from a *mathematically rigorous* standpoint have remained rather scarce. A reader looking for an introduction to its more advanced mathematical aspects must often either consult highly specialized monographs or search through numerous research articles available in peer-reviewed journals. It might even appear as if the mastery of certain techniques has survived from one generation of researchers to the next only by means of oral communication, through the use of chalk and blackboard...

It seems a general opinion that pedagogical introductory mathematically rigorous textbooks simply do not exist. This book aims at starting to bridge this gap. Both authors graduated in physics before turning to mathematical physics. As such, we have witnessed this lack from the student's point of view, before experiencing it, a few years later, from the teacher's point of view. Above all, this text aims to provide the material we would have liked to have had at our disposal when entering this field.

Although we hope that it will also be of interest to students in theoretical physics, this is in fact a book on *mathematical physics*. There is no general consensus on what this term actually refers to. In rough terms, what it means for us is: the analysis of problems originating in physics, at the level of rigor associated with mathematics. This includes the introduction of concepts and the development of tools enabling such an analysis. It is unfortunate that mathematical physics is often held in rather low esteem by physicists, many of whom see it as useless nitpicking and as dealing mainly with problems that they consider to be already fully understood. There are, however, very good reasons for these investigations. First, such an approach allows a very clear separation between the assumptions (the basic principles of the underlying theory, as well as the particulars of the model analyzed) and the actual derivation: once the proper framework is set, the entire analysis is done without further assumptions or approximations. This is essential in order to ensure that the phenomenon that has been derived is indeed a consequence of the starting hypotheses and not an artifact of the approximations made along the way. Second, to provide a complete mathematical analysis requires us to understand the phenomenon of interest in a much deeper and detailed way. In particular, it forces one to provide precise definitions and statements. This is highly useful in clarifying issues that are sometimes puzzling for students and, occasionally, researchers.

Let us emphasize two central features of this work.

- The first has to do with content. Equilibrium statistical mechanics has become such a rich and diverse subject that it is impossible to cover more than a fraction of it in a single book. Since our driving motivation is to provide an easily accessible introduction in a form suitable for self-study, our first decision was to focus on some of the most important and relevant examples rather than to present the theory from a broad point of view. We hope that this will help the reader build the necessary intuition, in concrete situations, as well as provide background and motivation for the general theory. We also refrained from introducing abstractions for their own sake and have done our best to keep the technical level as low as possible.
- The second central feature of this book is related to our belief that the main value of the proof of a theorem is measured by the extent to which it enhances understanding of the phenomena under consideration. As a matter of fact, the concepts and methods introduced in the course of a proof are often at least as important as the claim of the theorem itself. The most useful proof, for a beginner, is thus not necessarily the shortest or the most elegant one. For these reasons, we have strived to provide, throughout the book, the arguments we personally consider the most enlightening in the most simple manner possible.

These two features have shaped the book from its very first versions. (They have also contributed, admittedly, to the lengthiness of some chapters.) Together with the numerous illustrations and exercises, we hope that they will help the beginner to become familiar with some of the central concepts and methods that lie at the core of statistical mechanics.

As underlined by many authors, one of the main purposes of writing a book should be one's own pleasure. Indeed, leading this project to its conclusion was by and large a very enjoyable albeit long journey! But, beyond that, the positive feedback we have already received from students, and from colleagues who have used early drafts in their lectures, indicates that it may yet reach its goal, which is to help beginners enter this beautiful field.

Acknowledgements. This book benefited both directly and indirectly from the help and support of many colleagues. First and foremost, we would like to thank Charles Pfister who, as a PhD advisor, introduced both authors to this field of research many years ago. We have also learned much of what we know from our various co-authors during the past two decades. In particular, YV would like to express his thanks to Dima Ioffe, for a long, fruitful and very enjoyable ongoing collaboration.

Our warmest thanks also go to Aernout van Enter, who has been a constant source of support and feedback and whose enthusiasm for this project has always been highly appreciated!

We are very grateful to all the people who called to our attention various errors they found in preliminary versions of the book – in particular, Costanza Benassi, Quentin Berthet, Tecla Cardilli, Loren Coquille, Margherita Disertori, Hugo Duminil-Copin, Mauro Mariani, Philippe Moreillon, Sébastien Ott, Ron Peled, Sylvie Roelly, Costanza Rojas-Molina and Daniel Ueltschi.

We also thank Claudio Landim, Vladas Sidoravicius and Augusto Teixeira for their support and comments. Our warm thanks to Maria Eulalia Vares for her constant encouragement since the earliest drafts of this work.

SF thanks the Departamento de Matemática of the Federal University of Minas Gerais for its long-term support, Hans-Jörg Ruppen (CMS, EPFL), as well as the Section de Mathématiques of the University of Geneva for hospitality and financial support on countless visits during which large parts of this work were written. Both authors are also grateful to the Swiss National Science Foundation for its support, in particular through the NCCR SwissMAP.

Finally, writing this book would have been considerably less enjoyable without the following fantastic pieces of open source software: bash, GNU/Linux (openSUSE and Ubuntu flavors), GCC, GIMP, git, GNOME, gnuplot, Inkscape, KDE, Kile, LaTeX 2_ε, PGF/Tikz, POV-Ray, Processing, Python, Sketch (for LaTeX), TeXstudio, Vim and Xfig.

Sacha Friedli
Yvan Velenik

Conventions

$a \stackrel{\text{def}}{=} b$	a is defined as being b
$\mathbb{R}^d$	d-dimensional Euclidean space
$\mathbb{Z}^d$	d-dimensional cubic lattice
$\mathbb{R}_{\geq 0}$	nonnegative real numbers
$\mathbb{R}_{>0}$	positive real numbers
$\mathbb{Z}_{\geq 0}$	nonnegative integers: $0, 1, 2, 3, \ldots$
$\mathbb{N}, \mathbb{Z}_{>0}$	positive integers: $1, 2, 3, \ldots$
i	$\sqrt{-1}$
$\mathfrak{Re}\, z$, $\mathfrak{Im}\, z$	real and imaginary parts of $z \in \mathbb{C}$
$a \wedge b$	minimum of a and b
$a \vee b$	maximum of a and b
$\log$	natural logarithm, that is, in base $e = 2.718\ldots$
$A \subset B$	A is a (not necessarily proper) subset of B
$A \subsetneq B$	A is a proper subset of B
$A \triangle B$	symmetric difference
$\#A$, $\|A\|$	number of elements in the set A (if A is finite). At several places, also used to denote the Lebesgue measure.
$\delta_{m,n}$	Kronecker symbol: $\delta_{m,n} = 1$ if $m = n$, 0 otherwise
δ_x	Dirac measure at x: $\delta_x(A) = 1$ if $x \in A$, 0 otherwise
$\lfloor x \rfloor$	largest integer smaller than or equal to x
$\lceil x \rceil$	smallest integer larger than or equal to x

Asymptotic equivalence of functions will follow the standard conventions. For functions f, g, defined in the neighborhood of x_0 (possibly $x_0 = \infty$),

$f(x) \sim g(x)$	means $\lim_{x \to x_0} \frac{\log f(x)}{\log g(x)} = 1$,
$f(x) \simeq g(x)$	means $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$,
$f(x) \approx g(x)$	means $0 < \liminf_{x \to x_0} \frac{f(x)}{g(x)} \leq \limsup_{x \to x_0} \frac{f(x)}{g(x)} < \infty$,
$f(x) = O(g(x))$	means $\limsup_{x \to x_0} \left\lvert \frac{f(x)}{g(x)} \right\rvert < \infty$,
$f(x) = o(g(x))$	means $\lim_{x \to x_0} \left\lvert \frac{f(x)}{g(x)} \right\rvert = 0$.

As usual, A^B is identified with the set of all maps $f\colon B \to A$. A sequence of elements $a_n \in E$ will usually be denoted as $(a_n)_{n \geq 1} \subset E$. In several places, we will set

$0 \log 0 \stackrel{\text{def}}{=} 0$. Sums or products over empty families are defined as follows:

$$\sum_{i \in \varnothing} a_i \stackrel{\text{def}}{=} 0, \qquad \prod_{i \in \varnothing} a_i \stackrel{\text{def}}{=} 1.$$

Several important notations involving geometrical notions on $\mathbb{Z}^d$ will be defined at the end of the introduction and at the beginning of Chapter 3.

Contents