Trends in Abstract and Applied Analysis
Volume 6

The Strong Nonlinear Limit-Point/Limit-Circle Problem

Miroslav Bartušek
John R Graef

World Scientific

The Strong Nonlinear Limit-Point/ Limit-Circle Problem

The limit-point/limit-circle problem had its beginnings more than 100 years ago with the publication of Hermann Weyl's classic paper in *Mathematische Annalen* in 1910 on linear differential equations. This concept was extended to second-order nonlinear equations in the late 1970's and later, to higher order nonlinear equations. This monograph traces the development of what is known as the strong nonlinear limit-point and limit-circle properties of solutions. In addition to bringing together all such results into one place, some new directions that the study has taken as well as some open problems for future research are indicated.

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Miroslav Bartušek

Masaryk University, Czech Republic

John R Graef

University of Tennessee at Chattanooga, USA

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To Ivana and Frances, without whose help and support this work would never have been finished.

Preface

Since the publication of *The Nonlinear Limit-Point/Limit-Circle Problem*[1] (BDG), there have been a number of new developments in the study of this problem. First, the concept of nonlinear limit-point and nonlinear limit-circle solutions has been extended to equations with p-Laplacians. Secondly, the notions of strong nonlinear limit-point and strong nonlinear limit-circle solutions have been introduced and studied in some detail. The formulations of these ideas for equations of order greater than two and for delay differential equations have also been given.

It is our intention in this book to show the developments described above and then to indicate some new directions that the study of the limit-point/limit-circle problem has taken. Some open problems for future research are also indicated. This volume is not intended to be a sequel to (BDG) and in fact the present work can be read independently of (BDG). However, where appropriate, reference will be made to (BDG).

Chapter 1 gives an introduction to the limit-point/limit-circle problem from its origins with Hermann Weyl. Chapter 2 shows the development of the nonlinear limit-point/limit-circle from the results in the book of Bartušek, Došlá, Graef to the study of equations involving the p-Laplacian. In Chapter 3, the strong nonlinear limit-point and strong nonlinear limit-circle properties are examined beginning with Emden–Fowler equations and then progressing to equations of Emden–Fowler type with p-Laplacians. It is in this study that the terminology of "super-half-linear" and "sub-half-linear" equations is introduced. Generalized Thomas–Fermi equations and equations with forcing terms are also discussed. Second-order equations

[1] M. Bartušek, Z. Došlá, and J. R. Graef, Birkhäuser, Boston, 2004.

with damping terms are considered in Chapter 4 and higher order equations are studied in Chapter 5. Second-order delay differential equations are considered in Chapters 6 and 7. Chapter 8 talks about the use of transformations in studying the limit-point/limit-circle problem. The final chapter contains some open problems of varying degrees of difficulty and looks at extensions of the limit-point/limit-circle problem to different settings.

As was the case with the book (BDG), the connection between the nonlinear limit-point/limit-circle problem and other asymptotic properties of solutions such as boundedness, oscillation, and convergence to zero are interwoven throughout the discussion here.

Contents

Chapter 1

The Origins of the Limit-Point/Limit-Circle Problem

1.1 Introduction

In this chapter, we give a brief discussion of the origins of the limit-point/limit-circle problem including the reason for the choice of the terminology. This also prepares the reader for what is to appear in subsequent chapters.

1.2 The Weyl Alternative

The limit-point/limit-circle problem had its beginnings more than 105 years ago with the publication of Hermann Weyl's classic paper *Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörige Entwicklung willkürlicher Funktionen*[1] on eigenvalue problems for second-order linear differential equations. Weyl considered equations of the form

$$(a(t)y')' + r(t)y = \lambda y, \quad t \in [0, \infty),\ \lambda \in \mathbb{C}, \tag{1.2.1}$$

and he classified this equation to be of the *limit-circle* type if every solution is square integrable, i.e., belongs to L^2, and to be of the *limit-point* type if at least one solution does not belong to L^2. In the years that followed, there has been a great deal of work done on this problem due to its important relationship to the solution of certain boundary value problems; see, for example, Titchmarsh [160, 161].

[1] H. Weyl, *Math. Ann.* **68** (1910), 220–269.

Weyl's "limit-point/limit-circle" terminology comes from the proof of one of his fundamental results.

Theorem 1.2.1. *If* Im $\lambda \neq 0$, *then* (1.2.1) *always has a solution* $y \in L^2(\mathbb{R}_+)$, *i.e.*,

$$\int_0^\infty |y(t)|^2 dt < \infty.$$

Proof. For λ with $\operatorname{Im}\lambda \neq 0$, let φ and ψ be two linearly independent solutions of (1.2.1) satisfying the initial conditions

$$\varphi(0,\lambda) = 1, \quad \psi(0,\lambda) = 0,$$
$$\varphi'(0,\lambda) = 0, \quad \psi'(0,\lambda) = 1.$$

Now the functions $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ are analytic in λ on $\mathbb{C}$, and any other solution y is a linear combination of these two solutions, say,

$$y(t,\lambda) = \varphi(t,\lambda) + m(\lambda)\psi(t,\lambda),$$

where $m(\lambda)$ is to be determined. Choosing $b > 0$ and letting c_1 and c_2 be arbitrary but fixed constants, we need to find $m(\lambda)$ so that y satisfies

$$c_1 y(b,\lambda) + c_2 y'(b,\lambda) = 0. \tag{1.2.2}$$

The value of m depends on λ, b, c_1, and c_2, and moreover, has the form of the linear fractional transformation

$$m = \frac{Az + B}{Cz + D}.$$

The image of the real axis in the *z-plane* is a circle $\mathcal{C}_b$ in the *m-plane*. The solution y will satisfy (1.2.2) if and only if m is on $\mathcal{C}_b$. Using Green's identity, it can be shown that this is true if and only if

$$\int_0^b |y(s)|^2 ds = \frac{\operatorname{Im} m}{\operatorname{Im}\lambda},$$

and the radius of the circle $\mathcal{C}_b$ is

$$r_b = \left(2\operatorname{Im}\lambda \int_0^b |y(s)|^2 ds\right)^{-1}. \tag{1.2.3}$$

Now if $b_1 < b$, then

$$\int_0^{b_1} |y(s)|^2 ds < \int_0^b |y(s)|^2 ds,$$