Dimension Theory

RYSZARD ENGELKING

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Warsaw University

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PREFACE

Dimension theory is a branch of topology devoted to the definition and study of the notion of dimension in certain classes of topological spaces. It originated in the early twenties and rapidly developed during the next fifteen years. The investigations of that period were concentrated almost exclusively on separable metric spaces; they are brilliantly recapitulated in Hurewicz and Wallman's book *Dimension Theory*, published in 1941. After the initial impetus, dimension theory was at a standstill for ten years or more. A fresh start was made at the beginning of the fifties, when it was discovered that many results obtained for separable metric spaces can be extended to larger classes of spaces, provided that the dimension is properly defined. The last reservation necessitates an explanation. It is possible to define the dimension of a topological space X in three different ways, the small inductive dimension $\operatorname{ind}X$, the large inductive dimension $\operatorname{Ind}X$, and the covering dimension $\dim X$. The three dimension functions coincide in the class of separable metric spaces, i.e., $\operatorname{ind}X = \operatorname{Ind}X = \dim X$ for every separable metric space X. In larger classes of spaces the dimensions ind, Ind, and dim diverge. At first, the small inductive dimension ind was chiefly used; this notion has a great intuitive appeal and leads quickly and economically to an elegant theory. The dimension functions Ind and dim played an auxiliary role and often were not even explicitly defined. To attain the next stage of development of dimension theory, namely its extension to larger classes of spaces, first and foremost to the class of metrizable spaces, it was necessary to realize that in fact there are three theories of dimension and to decide which is the proper one. The adoption of such a point of view immediately led to the understanding that the dimension ind is practically of no importance outside the class of separable metric spaces and that the dimension dim prevails over the dimension Ind. The greatest achievement in dimension theory during the fifties was the discovery that $\operatorname{Ind}X = \dim X$ for every metric space X and the creation of a satisfactory dimension theory for metrizable spaces. Since that time many important results on dimension of topological spaces have been obtained; they primarily bear upon the covering dimension dim. Included among them are theorems of an entirely new type, such as the factorization theorems, with no counterpart in the classical theory, and a few quite complicated examples, which finally demarcated the range of applicability of various dimension functions.

The above outline of the history of dimension theory helps to explain the choice and arrangement of the material in the present book. In Chapter 1, which in itself constitutes more than half of the book, the classical dimension theory of separable metric spaces is developed. The purpose of the chapter is twofold: to present a self-contained exposition of the most important section of dimension theory and to provide the necessary geometric background for the rather abstract considerations of subsequent chapters. Chapters 2 and 3 are devoted to the large inductive dimension and the covering dimension, respectively. They contain the most significant results in dimension theory of general topological spaces and exhaustive information on further results. Chapter 4, the last in the book, develops the dimension theory of metrizable spaces. The interdependence of Chapters 2–4 is rather loose. After having read Chapter 1, the reader should be able to continue the reading according to his own interests or needs; in particular, he can read small fragments of Sections 3.1 and 3.2 and pass to Chapter 4 (cf. the introduction to that chapter).

Chapter 1 is quite elementary; the reader is assumed to be familiar only with the very fundamental notions of topology of separable metric spaces. The subsequent chapters are more difficult and demand from the reader some acquaintance with the notions and methods of general topology.

Each section ends with historical and bibliographic notes. Those are followed by problems which aim both at testing the reader's comprehension of the material and at providing additional information; the problems usually contain detailed hints, which, in fact, are outlines of proofs.

The mark □ indicates the end of a proof or of an example. If it appears immediately after the statement of a theorem, a proposition or a corollary, it means that the statement is obviously valid.

Numbers in square brackets refer to the bibliography at the end of the book. The papers of each author are numbered separately, the number being the year of publication. In referring to my *General Topology* (Engelking [1977]), which is quite often cited in the second half of the present book, the symbol [GT] is used.

In 1971–1973 I gave a two-year course of lectures on dimension theory at the Warsaw University; this book is based on the notes from those lectures. When preparing the present text, I availed myself of the comments of my students and colleagues. Thanks are due to K. Alster, J. Chaber, J. Kaniewski, P. Minc, R. Pol, T. Przymusiński, J. Przytycki and K. Wojtkowska. I am particularly obliged to Mrs. E. Pol, the first reader of this book, for her helpful cooperation, and to J. Krasinkiewicz for his careful reading of Chapter 1.

Ryszard Engelking

Warsaw, February 1977

CONTENTS

CHAPTER 1

DIMENSION THEORY OF SEPARABLE METRIC SPACES

In the present chapter the classical dimension theory of separable metric spaces is developed. Practically all the results of this chapter were obtained in the years 1920–1940. They constitute a canon on which, in subsequent years, dimension theory for larger classes of spaces was modelled. Similarly, in Chapters 2–4 we shall follow the pattern of Chapter 1 and constantly refer to the classical theory. This arrangement influences our exposition: the classical material is discussed here in relation to modern currents in the theory; in particular, the dimension functions Ind and dim are introduced at an early stage and are discussed simultaneously with the dimension function ind.

To avoid repetitions in subsequent chapters, a few definitions and theorems are stated in a more general setting, not for separable metric but for topological, Hausdorff, regular or normal spaces; this is done only where the generalization does not influence the proof. If the reader is not acquainted with the notions of general topology, he should read "metric space" instead of "topological space", "Hausdorff space", "regular space", and "normal space". Reading the chapter for the first time, one can omit Sections 1.4 and 1.12–1.14, which deal with rather special topics; similarly, the final parts of Sections 1.6, 1.8 and 1.9 can be skipped.

Let us describe briefly the contents of this chapter.

Section 1.1 opens with the definition of the small inductive dimension ind; in the sequel some simple consequences and reformulations of the definition are discussed. Sections 1.2 and 1.3 are devoted to a study of zero-dimensional spaces. We prove several important theorems, specified in the titles of the sections, which are generalized to spaces of higher dimension in Sections 1.5, 1.7 and 1.11.

In Section 1.4 we compare the properties of zero-dimensional spaces with the properties of different highly disconnected spaces. From this comparison it follows that the class of zero-dimensional spaces in the sense of the small inductive dimension is the best candidate for the zero level in a classification of separable metric spaces according to their dimension. The results of this section are not used in the sequel of the book.

Section 1.5 contains the first group of basic theorems on n-dimensional spaces. As will become clear further on, the theorems in this group depend on the dimension ind, whereas the theorems that follow them depend on the dimension dim. Besides the generalizations of five theorems proved in Sections 1.2 and 1.3 for zero-dimensional spaces, Section 1.5 contains the decomposition and addition theorems.

In Section 1.6 the large inductive dimension Ind and the covering dimension dim are introduced; they both coincide with the small inductive dimension ind in the class of separable metric spaces. In larger classes of spaces the dimensions ind, Ind and dim diverge. This subject will be discussed thoroughly in the following chapters. In particular, it will become evident that the dimension ind, though excellent in the class of separable metric spaces, loses its importance outside this class.

Section 1.7 opens with the compactification theorem. The location of this theorem at such an early stage in the exposition of dimension theory is a novelty which, it seems, permits a clearer arrangement of the material. From the compactification theorem the coincidence of ind, Ind, and dim for separable metric spaces is deduced.

In Section 1.8 we discuss the dimensional properties of Euclidean spaces. We begin with the fundamental theorem of dimension theory, which states that $\operatorname{ind} R^n = \operatorname{Ind} R^n = \dim R^n = n$; then we characterize n-dimensional subsets of R^n as sets with a non-empty interior, and we show that no closed subset of dimension $\leqslant n-2$ separates R^n. This last result is strengthened in Mazurkiewicz's theorem, which is established in the final part of the section with the assistance of Lebesgue's covering theorem.

Section 1.9 opens with the characterization of dimension in terms of extending mappings to spheres from a closed subspace over the whole space. From this characterization the Cantor-manifold theorem is deduced. In the final part of the section we give some information on the cohomological dimension.

In Section 1.10 we characterize n-dimensional spaces in terms of mappings with arbitrarily small fibers to polyhedra of geometric dimension $\leqslant n$ and develop the technics of nerves and $\varkappa$-mappings which are crucial for the considerations of this and the following section.

In Section 1.11 we prove that every n-dimensional space can be embedded in R^{2n+1} and we describe two subspaces of R^{2n+1} which contain topologically all n-dimensional spaces; the second of those is a compact space.

The last three sections are of a more special character. Section 1.12 is devoted to a study of the relations between the dimensions of the domain and the range of a continuous mapping. In Section 1.13 we characterize compact spaces of dimension $\leqslant n$ as spaces homeomorphic to the limits of inverse sequences of polyhedra of geometric dimension $\leqslant n$, and in Section 1.14 we briefly discuss the prospects for an axiomatization of dimension theory.

1.1. Definition of the small inductive dimension

1.1.1. Definition. To every regular space X one assigns the *small inductive dimension* of X, denoted by $\operatorname{ind} X$, which is an integer larger than or equal to -1 or the "infinite number" ∞; the definition of the dimension function ind consists in the following conditions:

(MU1) $\operatorname{ind} X = -1$ *if and only if* $X = \emptyset$;

(MU2) $\operatorname{ind} X \leqslant n$, *where* $n = 0, 1, \ldots$, *if for every point* $x \in X$ *and each neighbourhood* $V \subset X$ *of the point* x *there exists an open set* $U \subset X$ *such that*

$$x \in U \subset V \quad \textit{and} \quad \operatorname{ind} \operatorname{Fr} U \leqslant n-1;$$

(MU3) $\operatorname{ind} X = n$ *if* $\operatorname{ind} X \leqslant n$ *and* $\operatorname{ind} X > n-1$, *i.e., the inequality* $\operatorname{ind} X \leqslant n-1$ *does not hold;*

(MU4) $\operatorname{ind} X = \infty$ *if* $\operatorname{ind} X > n$ *for* $n = -1, 0, 1, \ldots$

The small inductive dimension ind is also called the *Menger–Urysohn dimension.*

Applying induction with respect to $\operatorname{ind} X$, one can easily verify that whenever regular spaces X and Y are homeomorphic, then $\operatorname{ind} X = \operatorname{ind} Y$, i.e., the small inductive dimension is a topological invariant.

In order to simplify the statements of certain results proved in the sequel, we shall assume that the formulas $n \leqslant \infty$ and $n+\infty = \infty+n = \infty+\infty = \infty$ hold for every integer n.

Since every subspace M of a regular space X is itself regular, if the dimension ind is defined for a space X it is also defined for every subspace M of the space X.

1.1.2. The subspace theorem. *For every subspace M of a regular space X we have* $\operatorname{ind} M \leqslant \operatorname{ind} X$.

Proof. The theorem is obvious if $\operatorname{ind} X = \infty$, so that one can suppose that $\operatorname{ind} X < \infty$. We shall apply induction with respect to $\operatorname{ind} X$. Clearly, the inequality holds if $\operatorname{ind} X = -1$.

Assume that the theorem is proved for all regular spaces whose dimension does not exceed $n-1 \geqslant -1$. Consider a regular space X with $\operatorname{ind} X = n$, a subspace M of the space X, a point $x \in M$ and a neighbourhood V of the point x in M. By the definition of the subspace topology, there exists an open subset V_1 of the space X satisfying the equality $V = M \cap V_1$. Since $\operatorname{ind} X \leqslant n$, there exists an open set $U_1 \subset X$ such that

$$x \in U_1 \subset V_1 \quad \text{and} \quad \operatorname{ind} \operatorname{Fr} U_1 \leqslant n-1.$$

The intersection $U = M \cap U_1$ is open in M and satisfies $x \in U \subset V$. The boundary $\operatorname{Fr}_M U$ of the set U in the space M is equal to $M \cap \overline{M \cap U_1} \cap \cap \overline{M \setminus U_1}$, where the bar denotes the closure operation in the space X; thus the boundary $\operatorname{Fr}_M U$ is a subspace of the space $\operatorname{Fr} U_1$. Hence, by the inductive assumption, $\operatorname{ind} \operatorname{Fr}_M U \leqslant n-1$, which—together with (MU2)—yields the inequality $\operatorname{ind} M \leqslant n = \operatorname{ind} X$. □

Sometimes it is more convenient to apply condition (MU2) in a slightly different form, involving the notion of a partition.

1.1.3. Definition. Let X be a topological space and A, B a pair of disjoint subsets of the space X; we say that a set $L \subset X$ is a *partition between A and B* if there exist open sets $U, W \subset X$ satisfying the conditions

(1) $\quad A \subset U, \quad B \subset W, \quad U \cap W = \emptyset \quad \text{and} \quad X \setminus L = U \cup W.$

Clearly, the partition L is a closed subset of X.

The notion of a partition is related to the notion of a separator. Let us recall that a set $T \subset X$ is a *separator between A and B*, or *T separates the space X between A and B*, if there exist two sets U_0 and W_0 open in the subspace $X \setminus T$ and such that $A \subset U_0$, $B \subset W_0$, $U_0 \cap W_0 = \emptyset$ and $X \setminus C = U_0 \cup V_0$. Obviously, a set $L \subset X$ is a partition between A and B if and only if L is a closed subset of X and L is a separator between A and B.

Separators are not to be confused with cuts, a related notion we will refer to in the notes below and in Section 1.8. Let us recall that a set $T \subset X$ is a *cut between A and B*, or *T cuts the space X between A and B*, if the sets A, B and T are pairwise disjoint and every continuum, i.e., a compact connected space $C \subset X$, intersecting both A and B intersects the set T. Clearly, every separator between A and B is a cut between A and B, but the two notions are not equivalent (see Problems 1.1.D and 1.8.F).

1.1.4. Proposition. *A regular space X satisfies the inequality* $\operatorname{ind} X \leqslant n \geqslant 0$ *if and only if for every point* $x \in X$ *and each closed set* $B \subset X$ *such that* $x \notin B$ *there exists a partition L between x and B such that* $\operatorname{ind} L \leqslant n-1$.

Proof. Let X be a regular space satisfying $\operatorname{ind} X \leqslant n \geqslant 0$; consider a point $x \in X$ and a closed set $B \subset X$ such that $x \notin B$. There exist a neighbourhood $V \subset X$ of the point x such that $\overline{V} \subset X \setminus B$ and an open set $U \subset X$ such that $x \in U \subset V$ and $\operatorname{ind} \operatorname{Fr} U \leqslant n-1$. One easily sees that the set $L = \operatorname{Fr} U$ is a partition between x and B; the sets U and $W = X \setminus \overline{U}$ satisfy conditions (1).

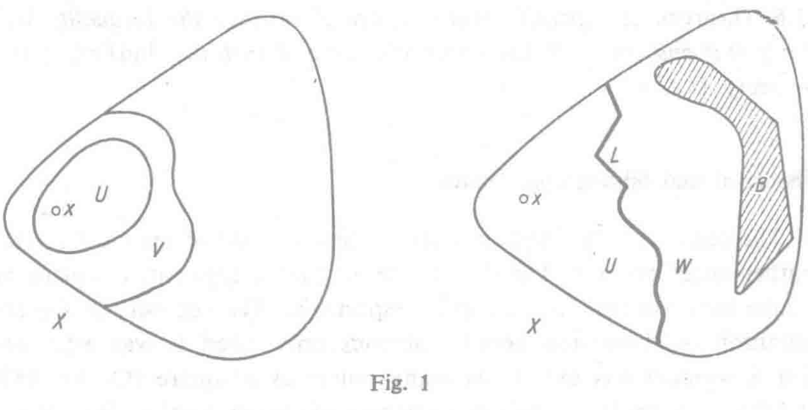

Fig. 1

Now, assume that a regular space X satisfies the condition of the theorem; consider a point $x \in X$ and a neighbourhood $V \subset X$ of the point x. Let L be a partition between x and $B = X \setminus V$ such that $\operatorname{ind} L \leqslant n-1$ and let $U, W \subset X$ be open subsets of X satisfying conditions (1). We have

$$x \in U \subset X \setminus W \subset X \setminus B = V$$

and

$$\operatorname{Fr} U \subset (X \setminus U) \cap (X \setminus W) = X \setminus (U \cup W) = L,$$

so that $\operatorname{ind} \operatorname{Fr} U \leqslant n-1$ by virtue of 1.1.2. Hence $\operatorname{ind} X \leqslant n$.□

Obviously, a regular space X satisfies the inequality $\operatorname{ind} X \leqslant n \geqslant 0$ if and only if X has a base $\mathscr{B}$ such that $\operatorname{ind} \operatorname{Fr} U \leqslant n-1$ for every $U \in \mathscr{B}$. In the realm of separable metric spaces this observation can be made more precise.

1.1.5. Lemma. *If a topological space X has a countable base, then every base $\mathscr{B}$ for the space X contains a countable family $\mathscr{B}_0$ which is a base for X.*

Proof. Let $\mathscr{D} = \{V_i\}_{i=1}^{\infty}$ be a countable base for the space X. For $i = 1, 2, \ldots$ define

$$\mathscr{B}_i = \{U \in \mathscr{B} \colon U \subset V_i\};$$

as $\mathscr{B}$ is a base for X, we have $\bigcup \mathscr{B}_i = V_i$. The subspace V_i of the space X also has a countable base, so that the open cover $\mathscr{B}_i$ of V_i contains a countable subcover $\mathscr{B}_{0,i}$. The family $\mathscr{B}_0 = \bigcup_{i=1}^{\infty} \mathscr{B}_{0,i} \subset \mathscr{B}$ is countable and is a base for X; indeed, every non-empty open subset of X can be represented as the union of a subfamily of $\mathscr{D}$, and thus can also be represented as the union of a subfamily of $\mathscr{B}_0$. $\square$

1.1.6. Theorem. *A separable metric space X satisfies the inequality* $\operatorname{ind} X \leqslant n \geqslant 0$ *if and only if X has a countable base $\mathscr{B}$ such that* $\operatorname{ind} \operatorname{Fr} U \leqslant n-1$ *for every $U \in \mathscr{B}$.* $\square$

Historical and bibliographic notes

The dimension of simple geometric objects is one of the most intuitive mathematical notions. There is no doubt that a segment, a square and a cube have dimension 1, 2 and 3, respectively. The necessity of a precise definition of dimension became obvious only when it was established that a segment has exactly as many points as a square (Cantor 1878), and that a square has a continuous parametric representation on a segment, i.e., that there exist continuous functions $x(t)$ and $y(t)$ such that points of the form $(x(t), y(t))$ fill out a square when t runs through a segment (Peano 1890). First and foremost the question arose whether there exists a parametric representation of a square on a segment which is at the same time one-to-one and continuous, i.e., whether a segment and a square are homeomorphic, and—more generally—whether the n-cube I^n and the m-cube I^m are homeomorphic if $n \neq m$; clearly, a negative answer was expected. Between 1890 and 1910 a few faulty proofs of the fact that I^n

and I^m are not homeomorphic if $n \neq m$ were produced and it was established that I, I^2 and I^3 are all topologically different.

The theorem that I^n and I^m are not homeomorphic if $n \neq m$ was proved by Brouwer in [1911]. The idea suggests itself that to prove this theorem one should define a function d assigning to every space a natural number, expressing the dimension of that space, such that to every pair of homeomorphic spaces the same natural number is assigned and that $d(I^n) = n$. It was none too easy, however, to discover such functions; the search for them gave rise to dimension theory. In Brouwer's paper [1911] no function d is explicitly defined, yet an analysis of the proof shows that to differentiate I^n and I^m for $n \neq m$ the author applies the fact that for a sufficiently small positive number ε it is impossible to transform the n-cube $I^n \subset R^n$ into a polyhedron $K \subset R^n$ of geometric dimension less than n by a continuous mapping $f\colon I^n \to K$ such that $\varrho(x, f(x)) < \varepsilon$ for every $x \in I^n$. As we shall show in Section 1.10, this property characterizes compact subspaces of R^n which have dimension equal to n. Another topological property of the n-cube I^n was discovered by Lebesgue in [1911], viz. the fact that I^n can be covered, for every $\varepsilon > 0$, by a finite family of closed sets with diameters less than ε such that all intersections of $n+2$ members of the family are empty, and cannot be covered by a finite family of closed sets with diameters less than 1 such that all intersections of $n+1$ members of the family are empty. Obviously, Lebesgue's observation implies that I^n and I^m are not homeomorphic if $n \neq m$. Though the proof outlined by Lebesgue contains a gap (filled by Brouwer in [1913] and by Lebesgue in [1921]), nevertheless the discovery of the new invariant was an important achievement which eventually led to the definition of the covering dimension. Lebesgue's paper [1911] contains one more important discovery. The author formulated the theorem (the proof was given in his paper [1921]) that for every continuous parametric representation $f(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ of the n-cube I^n on the closed unit interval I, some fibres of f, i.e., inverse images of one-point sets, have cardinality at least $n+1$, and that I^n has a continuous parametric representation on I with fibres of cardinality at most $n+1$.

A decisive step towards the definition of dimension was made by Poincaré in [1912], where he observed that the dimension is related to the notion of separation and could be defined inductively. Poincaré called attention to the simple fact that solids can be separated by surfaces, surfaces by lines, and lines by points. It was due to the character of the journal for which Poincaré was writing and also to his death in the same year

1912 that Poincaré's important ideas were not presented as a precise definition of dimension.

The first definition of a dimension function was given by Brouwer in [1913], where he defined a topological invariant of compact metric spaces, called *Dimensionsgrad*, and proved that the *Dimensionsgrad* of the n-cube I^n is equal to n. In conformity with Poincaré's suggestion, the definition is inductive and refers to the notion of a cut: Brouwer defined the spaces with *Dimensionsgrad* 0 as spaces which do not contain any continuum of cardinality larger than one (i.e., as punctiform spaces; cf. Section 1.4), and stated that a space X has *Dimensionsgrad* less than or equal to $n \geqslant 1$ if for every pair A, B of disjoint closed subsets of X there exists a closed set $L \subset X$ which cuts X between A and B and has *Dimensionsgrad* less than or equal to $n-1$. Brouwer's notion of dimension is not equivalent to what we now understand by the dimension of a compact metric space; however, the two notions coincide in the realm of locally connected compact metric spaces (the proof is based on the fact that in this class of spaces the notions of a separator and a cut are equivalent for closed subsets; cf. Kuratowski [1968], p. 258). Brouwer did not study the new invariant closely: he only used it to give another proof that I^n and I^m are not homeomorphic if $n \neq m$.

Referring to the second part of Lebesgue's paper [1911], Mazurkiewicz proved in [1915] that for every continuous parametric representation of the square I^2 on the interval I, some fibres of f have cardinality at least 3, and showed that every continuum $C \subset R^2$ whose interior in R^2 is empty can be represented as a continuous image of the Cantor set under a mapping with fibres of cardinality at most 2. These results led him to define the dimension of a compact metric space X as the smallest integer n with the property that the space X can be represented as a continuous image of a closed subspace of the Cantor set under a mapping f such that $|f^{-1}(x)| \leqslant n+1$ for every $x \in X$. As was proved later (cf. Problem 1.7.D), this definition is equivalent to the definition of the small inductive dimension, but Mazurkiewicz's paper had no influence on the development of dimension theory.

The definition of the small inductive dimension ind was formulated by Urysohn in [1922] and by Menger in [1923], both papers contain also Theorem 1.1.2. Menger and Urysohn, working independently, built the framework of the dimension theory of compact metric spaces, but Urysohn was ahead of Menger by a few months and was able to establish a larger number of basic properties of dimension. Urysohn's results are presented in a two-part paper, [1925] and [1926], published after the author's death in 1924, whereas Menger's results are contained in his papers [1923] and [1924] and in his book [1928]. A generalization of dimension theory to separable metric spaces is due to Tumarkin ([1925] and [1926]) and Hurewicz ([1927] and [1927b]). In [1927] Hurewicz, in a particularly successful way, made use of the inductive character of dimension and greatly simplified the proofs of some important theorems, e.g., the sum theorem and the decomposition theorem. Moreover, owing to his discovery of the compactification theorem, Hurewicz reduced, in a sense, the dimension theory of separable metric spaces to the dimension theory of compact metric spaces.

When the work of Menger and Urysohn drew the attention of mathematicians to the notion of dimension, Brouwer (in [1923], [1924], [1924a] and [1924b]) ascertained that the definition of his *Dimensionsgrad* was marred by a clerical error and that it should read exactly as the definition of the large inductive dimension (see Section 1.6) and thus should lead to the same notion of dimension for compact metric spaces; he also commented that even the original faulty definition of *Dimensionsgrad* could serve as a basis for an equally good, although different, dimension theory. Brouwer's arguments do not seem quite convincing. After the publication of Menger's book [1928] a heated discussion arose between Brouwer ([1928]) and Menger ([1929a], [1930], [1933]) concerning priority in defining the notion of dimension; a good account of this discussion is contained in Freudenthal's notes in the second volume of Brouwer's *Collected Papers* (Brouwer [1976]). The history of the first years of dimension theory and, in particular, an evaluation of the contributions of Menger and Urysohn can be found in Alexandroff [1951].

Problems

1.1.A. Observe that a metric space X satisfies the inequality $\operatorname{ind} X \leqslant n \geqslant 0$ if and only if for every point $x \in X$ and each positive number ε there exists a neighbourhood $U \subset X$ of the point x such that $\delta(U) < \varepsilon$ and $\operatorname{ind} \operatorname{Fr} U \leqslant n-1$.

1.1.B. To every regular space X and every point $x \in X$ one assigns the *dimension of X at the point x*, denoted by $\operatorname{ind}_x X$, which is an integer larger than or equal to 0 or the infinite number ∞; the definition consists