国外数学名著系列

（影印版）18

E.Hairer G.Wanner

Solving Ordinary Differential Equations II

Stiff and Differential -Algebraic Problems

Second Edition

常微分方程的解法II

刚性与微分代数问题

（第二版）

科学出版社
www.sciencep.com

图字:01-2005-6742

图书在版编目(CIP)数据

常微分方程的解法Ⅱ:刚性与微分代数问题:第2版=Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems (Second Edition)/(瑞士)海尔(Hairer, E.)等著. 一影印版 一北京:科学出版社, 2006

(国外数学名著系列)

ISBN 7-03-016681-7

Ⅰ. 常… Ⅱ. 海… Ⅲ. ①常微分方程-数值计算-英文②刚性常微分方程-英文③微分代数-英文 Ⅳ. 0241.81

中国版本图书馆 CIP 数据核字(2005)第 154396 号

科学出版社出版
北京东黄城根北街 16 号
邮政编码: 100717
http://www.sciencep.com

中国科学院印刷厂印刷

科学出版社发行 各地新华书店经销

*

2006 年 1 月第 一 版 开本:B5(720×1000)
2006 年 1 月第一次印刷 印张:39 3/4
印数:1—2 500 字数:752 000

定价:88.00 元

(如有印装质量问题,我社负责调换〈科印〉)

《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格（Springer）出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

To Evi and Myriam

From the Preface to the First Edition

"Whatever regrets may be, we have done our best."
(Sir Ernest Shackleton, turning back on 9 January 1909 at $88°23'$ South.)

Brahms struggled for 20 years to write his first symphony. Compared to this, the 10 years we have been working on these two volumes may even appear short.

This second volume treats stiff differential equations and differential algebraic equations. It contains three chapters: Chapter IV on one-step (Runge-Kutta) methods for stiff problems, Chapter V on multistep methods for stiff problems, and Chapter VI on singular perturbation and differential-algebraic equations.

Each chapter is divided into sections. Usually the first sections of a chapter are of an introductory nature, explain numerical phenomena and exhibit numerical results. Investigations of a more theoretical nature are presented in the later sections of each chapter.

As in Volume I, the formulas, theorems, tables and figures are numbered consecutively in each section and indicate, in addition, the section number. In cross references to other chapters the (latin) chapter number is put first. References to the bibliography are again by "author" plus "year" in parentheses. The bibliography again contains only those papers which are discussed in the text and is in no way meant to be complete.

It is a pleasure to thank J. Butcher, G. Dahlquist, and S.P. Nørsett (coauthor of Volume I) for their interest in the subject and for the numerous discussions we had with them which greatly inspired our work. Special thanks go to the participants of our seminar in Geneva, in particular Ch. Lubich, A. Ostermann and M. Roche, where all the subjects of this book have been presented and discussed over the years. Much help in preparing the manuscript was given by J. Steinig, Ch. Lubich and A. Ostermann who read and re-read the whole text and made innumerable corrections and suggestions for improvement. We express our sincere gratitude to them. Many people have seen particular sections and made invaluable suggestions and remarks: M. Crouzeix, P. Deuflhard, K. Gustafsson, G. Hall, W. Hundsdorfer, L. Jay, R. Jeltsch, J.P. Kauthen, H. Kraaijevanger, R. März, and O. Nevanlinna. ... Several pictures were produced by our children Klaudia Wanner and Martin Hairer, the one by drawing the other by hacking.

The marvellous, perfect and never failing TEX program of D. Knuth allowed us to deliver a camera-ready manuscript to Springer Verlag, so that the book could be produced rapidly and at a reasonable price. We acknowledge with pleasure the numerous remarks of the planning and production group of Springer Verlag concerning fonts, style and other questions of elegance.

March, 1991 The Authors

Preface to the Second Edition

The preparation of the second edition allowed us to improve the first edition by rewriting many sections and by eliminating errors and misprints which have been discovered. In particular we have included new material on

- methods with extended stability (Chebyshev methods) (Sect. IV.2);
- improved computer codes and new numerical tests for one- and multistep methods (Sects. IV.10 and V.5);
- new results on properties of error growth functions (Sects. IV.11 and IV.12);
- quasilinear differential equations with state-dependent mass matrix (Sect. VI.6).

We have completely reorganized the chapter on differential-algebraic equations by including three new sections on

- index reduction methods (Sect. VII.2);
- half-explicit methods for index-2 systems (Sect. VII.6);
- symplectic methods for constrained Hamiltonian systems and backward error analysis on manifolds (Sect. VII.8).

Our sincere thanks go to many persons who have helped us with our work:

- all readers who kindly drew our attention to several errors and misprints in the first edition, in particular C. Bendtsen, R. Chan, P. Chartier, T. Eirola, L. Jay, P. Kaps, J.-P. Kauthen, P. Leone, S. Maset, B. Owren, and L.F. Shampine;
- those who read preliminary versions of the new parts of this edition for their invaluable suggestions: M. Arnold, J. Cash, D.J. Higham, P. Kunkel, Chr. Lubich, A. Medovikov, A. Murua, A. Ostermann, and J. Verwer.
- the staff of the Geneva computing center and of the mathematics library for their constant help;
- the planning and production group of Springer-Verlag for numerous suggestions on presentation and style.

All figures have been recomputed and printed, together with the text, in Postscript. All computations and text processings were done on the SUN workstations of the Mathematics Department of the University of Geneva.

April 1996 The Authors

Contents

Chapter IV. Stiff Problems – One-Step Methods

IV.1	**Examples of Stiff Equations**	2
	Chemical Reaction Systems	3
	Electrical Circuits	4
	Diffusion	6
	A "Stiff" Beam	8
	High Oscillations	11
	Exercises	11
IV.2	**Stability Analysis for Explicit RK Methods**	15
	Stability Analysis for Euler's Method	15
	Explicit Runge-Kutta Methods	16
	Extrapolation Methods	18
	Analysis of the Examples of IV.1	18
	Automatic Stiffness Detection	21
	Step-Control Stability	24
	A PI Step Size Control	28
	Stabilized Explicit Runge-Kutta Methods	31
	Exercises	37
IV.3	**Stability Function of Implicit RK-Methods**	40
	The Stability Function	40
	A-Stability	42
	L-Stability and $A(\alpha)$-Stability	44
	Numerical Results	46
	Stability Functions of Order $\geq s$	47
	Padé Approximations to the Exponential Function	48
	Exercises	49
IV.4	**Order Stars**	51
	Introduction	51
	Order and Stability for Rational Approximations	56
	Stability of Padé Approximations	58
	Comparing Stability Domains	58
	Rational Approximations with Real Poles	61
	The Real-Pole Sandwich	62
	Multiple Real-Pole Approximations	67
	Exercises	70
IV.5	**Construction of Implicit Runge-Kutta Methods**	71
	Gauss Methods	71
	Radau IA and Radau IIA Methods	72

Lobatto IIIA, IIIB and IIIC Methods 75
The W-Transformation 77
Construction of Implicit Runge-Kutta Methods................ 83
Stability Function 84
Positive Functions 86
Exercises 89

IV.6 Diagonally Implicit RK Methods 91
Order Conditions 91
Stiffly Accurate SDIRK Methods 92
The Stability Function 96
Multiple Real-Pole Approximations with $R(\infty)=0$ 98
Choice of Method 99
Exercises 100

IV.7 Rosenbrock-Type Methods 102
Derivation of the Method 102
Order Conditions 104
The Stability Function 108
Construction of Methods of Order 4 108
Higher Order Methods 111
Implementation of Rosenbrock-Type Methods 111
The "Hump" 113
Methods with Inexact Jacobian (W-Methods) 114
Exercises 117

IV.8 Implementation of Implicit Runge-Kutta Methods 118
Reformulation of the Nonlinear System 118
Simplified Newton Iterations 119
The Linear System 121
Step Size Selection 123
Implicit Differential Equations 127
An SDIRK-Code 128
SIRK-Methods 128
Exercises 130

IV.9 Extrapolation Methods 131
Extrapolation of Symmetric Methods 131
Smoothing 133
The Linearly Implicit Mid-Point Rule 134
Implicit and Linearly Implicit Euler Method 138
Implementation 139
Exercises 142

IV.10 Numerical Experiments 143
The Codes Used 143
Twelve Test Problems 144
Results and Discussion 152
Partitioning and Projection Methods 160
Exercises 165

IV.11 Contractivity for Linear Problems 167
Euclidean Norms (Theorem of von Neumann) 168
Error Growth Function for Linear Problems 169
Small Nonlinear Perturbations 172
Contractivity in $\|\cdot\|_\infty$ and $\|\cdot\|_1$ 175
Study of the Threshold Factor 176

Absolutely Monotonic Functions 178
Exercises 179

IV.12 B-Stability and Contractivity 180
One-Sided Lipschitz Condition 180
B-Stability and Algebraic Stability 181
Some Algebraically Stable IRK Methods 183
AN-Stability 184
Reducible Runge-Kutta Methods 187
The Equivalence Theorem for S-Irreducible Methods 188
Error Growth Function 193
Computation of $\varphi_B(x)$ 195
Exercises 199

IV.13 Positive Quadrature Formulas and B-Stable RK-Methods 201
Quadrature Formulas and Related Continued Fractions 201
Number of Positive Weights 203
Characterization of Positive Quadrature Formulas 205
Necessary Conditions for Algebraic Stability 206
Characterization of Algebraically Stable Methods 209
The "Equivalence" of A- and B-Stability 211
Exercises 213

IV.14 Existence and Uniqueness of IRK Solutions 215
Existence 215
A Counterexample 217
Influence of Perturbations and Uniqueness 218
Computation of $\alpha_0(A^{-1})$ 220
Methods with Singular A 222
Lobatto IIIC Methods 223
Exercises 223

IV.15 B-Convergence 225
The Order Reduction Phenomenon 225
The Local Error 228
Error Propagation 229
B-Convergence for Variable Step Sizes 230
B-Convergence Implies Algebraic Stability 232
The Trapezoidal Rule 234
Order Reduction for Rosenbrock Methods 236
Exercises 237

Chapter V. Multistep Methods for Stiff Problems

V.1 Stability of Multistep Methods 240
The Stability Region 240
Adams Methods 242
Predictor-Corrector Schemes 244
Nyström Methods 245
BDF 246
The Second Dahlquist Barrier 247
Exercises 249

V.2 "Nearly" A-Stable Multistep Methods 250
$A(\alpha)$-Stability and Stiff Stability 250
High Order $A(\alpha)$-Stable Methods 251
Approximating Low Order Methods with High Order Ones 253

A Disc Theorem 254
Accuracy Barriers for Linear Multistep Methods 254
Exercises 259

V.3 Generalized Multistep Methods 261
Second Derivative Multistep Methods of Enright 261
Second Derivative BDF Methods 265
Blended Multistep Methods 266
Extended Multistep Methods of Cash 267
Multistep Collocation Methods 270
Methods of "Radau" Type 273
Exercises 275

V.4 Order Stars on Riemann Surfaces 279
Riemann Surfaces 279
Poles Representing Numerical Work 283
Order and Order Stars 284
The "Daniel and Moore Conjecture" 286
Methods with Property C 288
General Linear Methods 290
Dual Order Stars 295
Exercises 297

V.5 Experiments with Multistep Codes 300
The Codes Used 300
Exercises 304

V.6 One-Leg Methods and G-Stability 305
One-Leg (Multistep) Methods 305
Existence and Uniqueness 306
G-Stability 307
An Algebraic Criterion 309
The Equivalence of A-Stability and G-Stability 310
A Criterion for Positive Functions 313
Error Bounds for One-Leg Methods 314
Convergence of A-Stable Multistep Methods 317
Exercises 319

V.7 Convergence for Linear Problems 321
Difference Equations for the Global Error 321
The Kreiss Matrix Theorem 323
Some Applications of the Kreiss Matrix Theorem 326
Global Error for Prothero and Robinson Problem 328
Convergence for Linear Systems with Constant Coefficients 329
Matrix Valued Theorem of von Neumann 330
Discrete Variation of Constants Formula 332
Exercises 337

V.8 Convergence for Nonlinear Problems 339
Problems Satisfying a One-Sided Lipschitz Condition 339
Multiplier Technique 342
Multipliers and Nonlinearities 346
Discrete Variation of Constants and Perturbations 348
Convergence for Nonlinear Parabolic Problems 349
Exercises 354

V.9 Algebraic Stability of General Linear Methods 356
G-Stability 356

Algebraic Stability 357
AN-Stability and Equivalence Results 359
Multistep Runge-Kutta Methods 362
Simplifying Assumptions 363
Quadrature Formulas 365
Algebraically Stable Methods of Order $2s$ 366
B-Convergence 368
Exercises 370

Chapter VI. Singular Perturbation Problems and Index 1 Problems

VI.1 Solving Index 1 Problems 372
Asymptotic Solution of van der Pol's Equation 372
The ε-Embedding Method for Problems of Index 1 374
State Space Form Method 375
A Transistor Amplifier 376
Problems of the Form $Mu' = \varphi(u)$ 378
Convergence of Runge-Kutta Methods 380
Exercises 381

VI.2 Multistep Methods 382
Methods for Index 1 Problems 382
Convergence for Singular Perturbation Problems 383
Exercises 387

VI.3 Epsilon Expansions for Exact and RK Solutions 388
Expansion of the Smooth Solution 388
Expansions with Boundary Layer Terms 389
Estimation of the Remainder 391
Expansion of the Runge-Kutta Solution 392
Convergence of RK-Methods for Differential-Algebraic Systems 394
Existence and Uniqueness of the Runge-Kutta Solution 397
Influence of Perturbations 398
Estimation of the Remainder in the Numerical Solution 399
Numerical Confirmation 403
Perturbed Initial Values 405
Exercises 406

VI.4 Rosenbrock Methods 407
Definition of the Method 407
Derivatives of the Exact Solution 408
Trees and Elementary Differentials 409
Taylor Expansion of the Exact Solution 411
Taylor Expansion of the Numerical Solution 412
Order Conditions 415
Convergence 416
Stiffly Accurate Rosenbrock Methods 418
Construction of RODAS, a Stiffly Accurate Embedded Method.. 420
Inconsistent Initial Values 422
Exercises 424

VI.5 Extrapolation Methods 426
Linearly Implicit Euler Discretization 426
Perturbed Asymptotic Expansion 428
Order Tableau 431

Error Expansion for Singular Perturbation Problems 433
Dense Output 438
Exercises 441

VI.6 Quasilinear Problems 442
Example: Moving Finite Elements 442
Problems of Index One 445
Numerical Treatment of $C(y)y' = f(y)$ 446
Extrapolation Methods 447
Exercises 448

Chapter VII. Differential-Algebraic Equations of Higher Index

VII.1 The Index and Various Examples 452
Linear Equations with Constant Coefficients 452
Differentiation Index 454
Differential Equations on Manifolds 457
The Perturbation Index 459
Control Problems 461
Mechanical Systems 463
Exercises 465

VII.2 Index Reduction Methods 468
Index Reduction by Differentiation 468
Stabilization by Projection 470
Differential Equations with Invariants 472
Methods Based on Local State Space Forms 474
Overdetermined Differential-Algebraic Equations 477
Unstructured Higher Index Problems 478
Exercises 480

VII.3 Multistep Methods for Index 2 DAE 481
Existence and Uniqueness of Numerical Solution............. 482
Influence of Perturbations 484
The Local Error 485
Convergence for BDF 486
General Multistep Methods 489
Solution of the Nonlinear System by Simplified Newton 490
Exercises 491

VII.4 Runge-Kutta Methods for Index 2 DAE 492
The Nonlinear System 492
Estimation of the Local Error 494
Convergence for the y-Component 496
Convergence for the z-Component 497
Collocation Methods 498
Superconvergence of Collocation Methods 500
Projected Runge-Kutta Methods 502
Summary of Convergence Results 504
Exercises 505

VII.5 Order Conditions for Index 2 DAE 506
Derivatives of the Exact Solution 506
Trees and Elementary Differentials 507
Taylor Expansion of the Exact Solution 508

Derivatives of the Numerical Solution ... 510
Order Conditions ... 512
Simplifying Assumptions ... 514
Projected Runge-Kutta Methods ... 515
Exercises ... 518

VII.6 Half-Explicit Methods for Index 2 Systems ... 519
Half-Explicit Runge-Kutta Methods ... 520
Extrapolation Methods ... 525
β-Blocked Multistep Methods ... 527
Exercises ... 529

VII.7 Computation of Multibody Mechanisms ... 530
Description of the Model ... 530
Fortran Subroutines ... 533
Computation of Consistent Initial Values ... 535
Numerical Computations ... 536
A Stiff Mechanical System ... 541
Exercises ... 542

VII.8 Symplectic Methods for Constrained Hamiltonian Systems ... 543
Properties of the Exact Flow ... 544
First Order Symplectic Method ... 545
SHAKE and RATTLE ... 548
The Lobatto IIIA-IIIB Pair ... 550
Composition Methods ... 554
Backward Error Analysis (for ODEs) ... 555
Backward Error Analysis on Manifolds ... 559
Exercises ... 562

Appendix. Fortran Codes ... 565
Driver for the Code RADAU5 ... 566
Subroutine RADAU5 ... 568
Subroutine RADAUP ... 574
Subroutine RODAS ... 574
Subroutine SEULEX ... 575
Problems with Special Structure ... 575
Use of SOLOUT and of Dense Output ... 576

Bibliography ... 577

Symbol Index ... 605

Subject Index ... 607

Chapter IV. Stiff Problems – One-Step Methods

This chapter introduces stiff (styv (Swedish first!), steif (German), stìf (Islandic), stijf (Dutch), raide (French), rígido (Spanish), rígido (Portuguese), stiff (Italian), kankea (Finnish), δύσκαμπτο (Greek), merev (Hungarian), rigid (Rumanian), tog (Slovenian), čvrst (Serbo-Croatian), tuhỳ (Czecho-Slovakian), sztywny (Polish), jäik (Estonian), stiegrs (Latvian), standus (Lithuanian), stign (Breton), zurrun (Basque), sert (Turkish), жесткий (Russian), твърд (Bulgarian), קשיח (Hebrew), سَاق (Arabic), بِنُگِک (Urdu), سخت (Persian), कठिण (Sanscrit), कड़ा (Hindi), 刚性 (Chinese), 硬い (Japanese), cứng (Vietnamese), ngumu (Swaheli) ...) differential equations. While the intuitive meaning of stiff is clear to all specialists, much controversy is going on about it's correct mathematical definition (see e.g. p.360-363 of Aiken (1985)). The most pragmatical opinion is also historically the first one (Curtiss & Hirschfelder 1952): *stiff equations are equations where certain implicit methods, in particular BDF, perform better, usually tremendously better, than explicit ones.* The eigenvalues of the Jacobian $\partial f / \partial y$ play certainly a role in this decision, but quantities such as the dimension of the system, the smoothness of the solution or the integration interval are also important (Sections IV.1 and IV.2).

Stiff equations need new concepts of stability (A-stability, Sect. IV.3) and lead to mathematical theories on order restrictions (order stars, Sect. IV.4). Stiff equations require implicit methods; we therefore focus in Sections IV.5 and IV.6 on implicit Runge-Kutta methods, in IV.7 on (semi-implicit) Rosenbrock methods and in IV.9 on semi-implicit extrapolation methods. The actual efficient implementation of implicit Runge-Kutta methods poses a number of problems which are discussed in Sect. IV.8. Section IV.10 then reports on some numerical experience for all these methods.

With Sections IV.11, IV.12 and IV.13 we begin with the discussion of contractivity (B-stability) for linear and nonlinear differential equations. The chapter ends with questions of existence and numerical stability of the implicit Runge-Kutta solutions (Sect. IV.14) and a convergence theory which is independent of the stiffness (B-convergence, Sect. IV.15).

IV.1 Examples of Stiff Equations

... Around 1960, things became completely different and everyone became aware that the world was full of stiff problems.
(G. Dahlquist in Aiken 1985)

Stiff equations are problems for which explicit methods don't work. Curtiss & Hirschfelder (1952) explain stiffness on one-dimensional examples such as

$$y' = -50(y - \cos x). \tag{1.1}$$

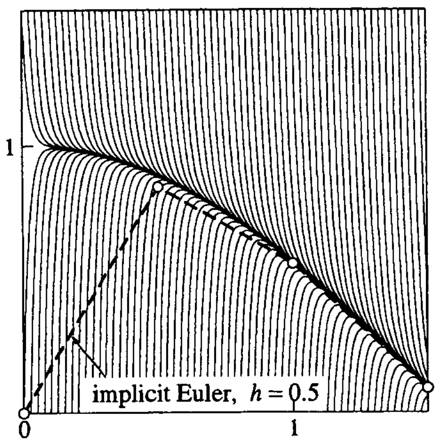

Fig. 1.1. Solution curves of (1.1) with implicit Euler solution

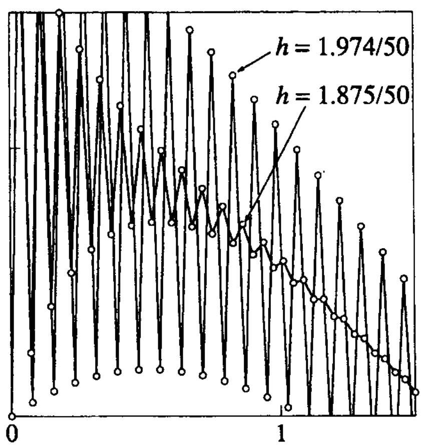

Fig. 1.2. Explicit Euler for $y(0) = 0$, $h = 1.974/50$ and $1.875/50$

Solution curves of Equation (1.1) are shown in Fig. 1.1. There is apparently a smooth solution in the vicinity of $y \approx \cos x$ and all other solutions reach this one after a rapid "transient phase". Such transients are typical of stiff equations, but are neither sufficient nor necessary. For example, the solution with initial value $y(0) = 1$ (more precisely $2500/2501$) has *no* transient. Fig. 1.2 shows Euler polygons for the initial value $y(0) = 0$ and step sizes $h = 1.974/50$ (38 steps) and $h = 1.875/50$ (40 steps). We observe that whenever the step size is a little too large (larger than 2/50), the numerical solution goes too far beyond the equilibrium and violent oscillations occur.

Looking for better methods for differential equations such as (1.1), Curtiss and Hirschfelder discovered the BDF method (see Sect. III.1): the approximation

$y \approx \cos x$ (i.e., $f(x,y)=0$) is only a crude approximation to the smooth solution, since the derivative of $\cos x$ is not zero. It is much better, for a given solution value y_n, to search for a point y_{n+1} where the slope of the vector field is directed towards y_n, hence

$$\frac{y_{n+1}-y_n}{h} = f(x_{n+1}, y_{n+1}). \tag{1.2}$$

This is the implicit Euler method. The dotted line in Fig. 1.1 consists of three implicit Euler steps and demonstrates impressively the good stability property of this method. Equation (1.1) is thus apparently "stiff" in the sense of Curtiss and Hirschfelder.

Extending the above idea "by taking higher order polynomials to fit y at a large number of points" then leads to the BDF methods.

Chemical Reaction Systems

> When the equations represent the behaviour of a system containing a number of fast and slow reactions, a forward integration of these equations becomes difficult. (H.H. Robertson 1966)

The following example of Robertson's (1966) has become very popular in numerical studies (Willoughby 1974):

$$\begin{array}{cccl} A & \xrightarrow{0.04} & B & \text{(slow)} \\ B+B & \xrightarrow{3\cdot 10^7} & C+B & \text{(very fast)} \\ B+C & \xrightarrow{10^4} & A+C & \text{(fast)} \end{array} \tag{1.3}$$

which leads to the equations

$$\begin{array}{llll} \text{A:} & y_1' = -0.04y_1 + 10^4 y_2 y_3 & & y_1(0)=1 \\ \text{B:} & y_2' = 0.04y_1 - 10^4 y_2 y_3 & -3\cdot 10^7 y_2^2 & y_2(0)=0 \\ \text{C:} & y_3' = & 3\cdot 10^7 y_2^2 & y_3(0)=0. \end{array} \tag{1.4}$$

After a bad experience with explicit Euler just before, let's try a higher order method and a more elaborate code for this example: DOPRI5 (cf. Volume 1). The numerical solutions obtained for y_2 with $Rtol = 10^{-2}$ (209 steps) as well as with $Rtol = 10^{-3}$ (205 steps) and $Atol = 10^{-6}\cdot Rtol$ are displayed in Fig. 1.3. Fig. 1.4 presents the step sizes used by the code and also the local error estimates. There, all rejected steps are crossed out.

We observe that the solution y_2 rapidly reaches a quasi-stationary position in the vicinity of $y_2' = 0$, which in the beginning ($y_1 = 1$, $y_3 = 0$) is at $0.04 \approx 3\cdot 10^7 y_2^2$, hence $y_2 \approx 3.65\cdot 10^{-5}$, and then very slowly goes back to zero again. The numerical method, however, integrates this smooth solution by thousands of apparently unnecessary steps. Moreover, the chosen step sizes are more or less independent of the chosen tolerance. Hence, they seem to be governed by stability