Huang Yisheng

BCI-Algebra

(BCI-代数)

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In memory of my master Professor Chen Zhaomu

Preface

This book is mainly designed for the graduate students who are interested in the theory of BCK and BCI-algebras.

BCI-algebras are a wider class than BCK-algebras, introduced by Kiyoshi Iséki in 1966. BCI-algebras as a class of logical algebras are the algebraic formulations of the set difference together with its properties in set theory and the implicational functor in logical systems. They are closely related to partially ordered commutative monoids as well as various logical algebras. Their names are originated from the combinators B, C, K and I in combinatory logic. The early research work was mainly carried out among the Japanese mathematicians Kiyoshi Iséki and Shotaro Tanaka, etc. who did a great deal of foundation work. Since late 1970s, their work has been paid much attention. In particular, the participation in the research of Polish mathematicians Tadeusz Traczyk and Andrzej Wroński as well as Australian mathematician William H. Cornish, etc. is making this branch of algebra develop rapidly. Many interesting and important results are discovered continuously. Now, the theory of BCI-algebras has been widely spread.

The structure of this book is similar to that of *Two B-Algebras*, the teaching materials by Zhaomu Chen. Some of the contents are drawn from the following two books: *BCK-Algebras* by Jie Meng and Young Bae Jun, and *An Introduction to BCI-Algebras* by Jie Meng and Yonglin Liu. Most contents come from firsthand information. Because Professor Huishi Li's axiom system is adopted and also because this book's system is required, many proofs are properly modified. This book is only an analysis on the general theoretical basis of BCI-algebras. Therefore the materials are somewhat limited. For example, p-semisimple algebras only take a little space, contents on the topology and category theories and fuzzy BCI-algebras are omitted. I think, it may be more proper to do so for an elementary book. We try what we can to use all kinds of notations and terminologies used by most papers' authors. More examples are given and the materials are handled more systematically and various arguments are written in more details so as to be read easily. Quite a lot of exercises are arranged at the end of every section. They are also the component part of our theory. The exercises with the sign $*$ are more difficult for the beginners who can leave them away. The two appendices at the back of the text will be of great value to those who are interested in further research of BCI-algebras.

For many times the late Professor Zhaomu Chen, my former teacher, encouraged me to compile this book. Professors Jie Meng, Hao Jiang, Young Bae Jun, Yonglin Liu and Doctor Eun Hwan Roh provided much valuable information. Professor Hao Jiang went over the manuscript and pointed out some mistakes. Miss Liying Chen in Longyan Teachers' College provided me much help in English expression. Mr Shenrong Lu in Longyan Teachers' College gave me much guidance in using computer. Mr Wenqing Zhang in Sanming College read through all pages and corrected some spelling and grammatical mistakes. Sanming College that I am working in now offers financial aid for publication of the book. Here, I extend my heartfelt thanks to all those who have supported, helped and encouraged me to write this book.

Huang Yisheng
Sanming, Fujian
December, 2003

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Chapter 0

Introduction

BCK-algebras and BCI-algebras are abbreviated to two B-algebras. The former was raised in 1966 by Y. Imai and K. Iséki, Japanese mathematicians, and the latter was put forward in the same year due to K. Iséki.

Two B-algebras are originated from two different sources. One of the motivations is based on set theory. In set theory, there are three most elementary and fundamental operations. They are the union, intersection and set difference. If we consider those three operations and their properties, then as a generalization of them, we have the notion of Boolean algebras. If we take both of the union and intersection, then as a general algebra, the notion of distributive lattices is obtained. Moreover, if we consider the union or the intersection alone, we have the notion of upper semilattices or lower semilattices. However, the set difference together with its properties had not been considered systematically before K. Iséki.

Another motivation is from propositional calculi. There are some systems which contain the only implicational functor among logical functors, such as the system of positive implicational calculus, the system of weak positive implicational calculus, BCK-system and BCI-system. Undoubtedly there are common properties among those systems.

We know very well that there are close relationships between the notions of the set difference in set theory and the implication functor in logical systems. For example, we have the following simple inclusion relations in set theory:

$$(A - B) - (A - C) \subseteq C - B,$$
$$A - (A - B) \subseteq B.$$

These are similar to the propositional formulas in propositional calculi:

$$(p \to q) \to ((q \to r) \to (p \to r)),$$
$$p \to ((p \to q) \to q).$$

It raises the following questions. What are the most essential and fundamental properties of these relationships? Can we formulate a general algebra from the above consideration? How will we find an axiom system to establish a good theory of general algebras? Answering these questions, K. Iséki formulated the notions of two B-algebras in which BCI-algebras are a wider class than BCK-algebras. Their names are taken from BCK and BCI-systems in combinatory logic.

§0.1 Mappings Abelian Groups Binary Relations

We begin our discussion with a brief survey of some fundamental notions which will be frequently mentioned.

A **mapping** $f : A \to B$ is a rule of correspondences from a nonempty set A to another set B, satisfying the condition that for any $a \in A$ there exists a unique element $b \in B$ such that a corresponds with b (symbolically, $f(a) = b$ or $f : a \mapsto b$), where A is called the **domain** of f, B the **codomain** of f, and the set, $\mathrm{Im}(f) = \{f(a) \mid a \in A\}$, the **image** of f. Also, we call b the **image** of a under f, and a an **inverse image** of b under f.

In general, an element b in B may have many inverse images, or may not have any one. If for all $b \in B$ there is at least an inverse image of b, i.e., $\mathrm{Im}(f) = B$, we call f a **surjection**. If for any $b \in \mathrm{Im}(f)$ there is one and only one inverse image of b, f is called an **injection**. Of course, a **bijection** $f : A \to B$ is a mapping which is both surjective and injective.

Denoting a^* for the image of a under a mapping $f : A \to B$, we can regard $*$ as an operation from A to B. From the above statements of mappings, an operation $*$ from A to B has to satisfy: ① uniqueness: the result a^* after a through the operation $*$ is unique; ② closeness: a^* must belong to B. For example, the power $a^{*_1} = a^2$ $(a \in Z)$ can be regarded as an operation $*_1$ from the set Z of all integers to itself, and $(a, b)^{*_2} = |ab|$ $(a, b \in Z)$ as an operation $*_2$ from the Cartesian product set $Z \times Z$ to Z, where $|\bullet|$ is the absolute value of $\bullet$.

Let A be a nonempty set. An operation $*$ from the Cartesian product set A^n to A is called an n-**ary operation** on A. Especially, a 2-ary operation is just a **binary operation**, and a 1-ary operation is a **unary operation**. Then the above operation $*_1$ is a unary operation on Z, and $*_2$ is a binary operation on Z.

There are some elements in a set, which play special roles. For example, 0 and 1 in Z have respectively the familiar properties: $x + 0 = x$ and $x \cdot 1 = x$ for all $x \in Z$. Such an element is actually a special mapping, and so in our point of view, it can be regarded as a so-called **nullary operation** (usually, it is called a **constant**).

A system consisting of a nonempty set A together with some operations on A and their laws is called an **algebra**. Those operations on A are usually described by the **type** of this algebra. For example, a group $(G; \cdot, e)$ is an algebra of type (2, 0). That is to say, this system consists of a nonempty set G and a binary operation $\cdot$ on G as well as a constant e (i.e., a nullary operation). Similarly, a ring R is an algebra of type (2, 2, 0), and a field F is of type (2, 2, 0, 0).

A nonempty subset B of an algebra A, which contains all constants of A if they exist, is called a **subalgebra** of A if B is closed under all operations on A and if all laws in A are still valid in B.

Abelian groups will play a basic role in BCI-algebras. We recall that an algebra $(G; \cdot, e)$ of type (2, 0) is said an **Abelian group** (or a **commutative group**) if the following hold:

(1) associative law: $(ab)c = a(bc)$ for any a, b, $c \in G$;

(2) commutative law: $ab = ba$ for any a, $b \in G$;

(3) the unit element of G exists: there is an element $e \in G$ such that $ea = a$ for any $a \in G$;

(4) every element in G is invertible: for any $a \in G$, there exists $b \in G$ such that $ab = e$.

Several simple examples of Abelian groups are as follows: the additive group of integers, the additive group of residue classes modulo n, the group of roots of unity.

We also recall that an algebra $(M; \cdot, e)$ of type (2, 0) is called a **monoid** if the operation $\cdot$ on M satisfies the associative law and the constant e is a unit element of M. Any group is obviously a monoid. A **sub-semigroup** S of a monoid M means that S is a nonempty subset of M and S is closed under the operation $\cdot$ on M. A **submonoid** of M is just a subalgebra of the monoid M as an algebra. A sub-semigroup is generally not a submonoid, for example, the set $\{1, 2, 3, \cdots\}$ of natural numbers is a sub-semigroup of the additive group $(Z; +, 0)$ of integers, but not a submonoid of it.

Because every element a in a group G has its inverse element a^{-1}, we can induce a binary operation $*$ on G by putting $a * b = a \cdot b^{-1}$. It is interesting that if a non-vacuous subset H of G is closed under $*$, it must be a subgroup of G. However, if H is closed under $\cdot$, it may not be a subgroup of G. From this, we see that the operation $*$ on G is sometimes more effective and useful than the operation $\cdot$ on G, although $*$ does not satisfy the associative and commutative laws.

Binary relations are a generalization of the notion of mappings. Roughly speaking, a binary relation is an assertion determining the correctness between two objects. We now describe this notion. Let A, B be two non-vacuous sets and let θ be an assertion between A and B. If each ordered pair (a, b) of elements $a \in A$ and $b \in B$ either fits or unfits the assertion θ, we call θ a **binary relation** between A and B. Especially, if $A = B$, we say the relation θ is on A. We denote $a \sim b\,(\theta)$ for a and b fitting the relation θ. In the viewpoint of abstract, a binary relation θ between A and B can be simply regarded as a subset of $A \times B$. In fact, we first note that $\{(a, b) \in A \times B \mid a \sim b\,(\theta)\}$ is evidently a subset of $A \times B$. Next, given a subset C of $A \times B$, we can provide a binary relation θ between A and B as follows: $a \sim b\,(\theta)$ if and only if $(a, b) \in C$.

Equivalence relations are an important class of binary relations. If a binary relation θ on A satisfies the following: for any $a, b, c \in A$,

(1) reflexivity: $a \sim a\,(\theta)$;
(2) symmetry: $a \sim b\,(\theta)$ implies $b \sim a\,(\theta)$;
(3) transitivity: $a \sim b\,(\theta)$ and $b \sim c\,(\theta)$ imply $a \sim c\,(\theta)$,

then we call it an **equivalence relation** on A. An interesting example of such relations is the congruence modulo n in number theory. In this case, we are used to denote $a \sim b\,(\theta)$ by $a \equiv b \pmod{n}$ in the sense that $a - b$ is a multiple of n.

A **partition** π of a set A means that π is a collection of non-vacuous subsets of A such that the union of all members in π is the whole of A and distinct members in π are disjoint. An equivalence relation can be characterized by a partition. In fact, if θ is an equivalence relation on A, then the **quotient set** $\pi = \{\overline{a} \mid a \in A\}$ determines a partition of A, where $\overline{a}$ is the set $\{x \in A \mid x \sim a\,(\theta)\}$, called the **equivalence class** containing the element a. Conversely, if π is a partition of A, then the following relation θ on A is an equivalence relation: $a \sim b\,(\theta)$ if and only if $a, b \in C$ for some $C \in \pi$.

Another important class of binary relations is partial orderings. For such a relation θ, the symbol $a \sim b\,(\theta)$ is usually written as $a \leqslant b$. A binary relation $\leqslant$ on a set A is called a **partial ordering** if the following hold: for any $a, b, c \in A$,

(1) reflexivity: $a \leqslant a$;
(2) anti-symmetry: $a \leqslant b$ and $b \leqslant a$ imply $a = b$;
(3) transitivity: $a \leqslant b$ and $b \leqslant c$ imply $a \leqslant c$.

A typical example of partial orderings is the inclusion relation $\subseteq$ of sets. If $\leqslant$ is a partial ordering on A, the system $(A; \leqslant)$ is said a **partially ordered set**. If we do have either $a \leqslant b$ or $b \leqslant a$ for any a, $b \in A$, we call such a partially ordered set $(A; \leqslant)$ a **totally ordered set**. Sometimes, we denote $a \leqslant b$ and $a \neq b$ by $a < b$. And we write $a \geqslant b$ as an alternative for $b \leqslant a$ and $a > b$ for $b < a$.

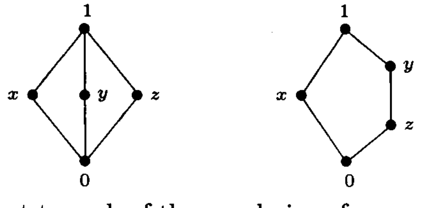
§0.2 Lattices Boolean Algebras

Given two elements a and b in a partially ordered set $(L; \leqslant)$, an element u in L is said a **lower bound** of a and b if $u \leqslant a$ and $u \leqslant b$. The element u is said a **greatest lower bound** of a and b if ① u is a lower bound of a and b; ② $v \leqslant u$ for every lower bound v of a and b. The greatest lower bound is clearly unique if it exists. In a similar fashion we can define an **upper bound** and the **least upper bound** of a and b. The greatest lower bound is often abbreviated to g.l.b., and the least upper bound to l.u.b. There are some partially ordered sets, each of which has the greatest element or the least element. Sometimes, we denote them by 1 and 0, called the **unit element** and the **zero element** respectively.

A partially ordered set $(L; \leqslant)$ is called a **lower semilattice** if any two elements in L have the greatest lower bound of them. It is called an **upper semilattice** if each pair of elements in L has its least upper bound. If $(L; \leqslant)$ is both a lower semilattice and an upper semilattice, we call it a **lattice**.

Let's list several examples of lattices as preliminaries. It has been known that the partial ordering of a partially ordered set of finite order can be described by a diagram, called a **Hasse diagram**.

Example 0.2.1. Let L be the set $\{x, y, z, 0, 1\}$. Define two partial orderings on L by the following Hasse diagrams respectively:



Then L with respect to each of these orderings forms a lattice. We call the former the **rhombus lattice**, and the latter the **pentagon lattice**.

We always denote $\subseteq$ for the inclusion relation of sets in this book. If A is properly contained in B, we will write it by $A \subset B$.

Example 0.2.2. (1) $(2^S; \subseteq)$ is a lattice, called the **power set lattice** of S, where 2^S is the power set of a set S (i.e., the collection of all subsets of S), and g.l.b.$\{A, B\} = A \cap B$, l.u.b.$\{A, B\} = A \cup B$ for any $A, B \in 2^S$.

(2) $(L(V); \subseteq)$ is a lattice, called the **subspace lattice** of V, where $L(V)$ is the collection of the whole subspaces of a vector space V over a field, and g.l.b.$\{A, B\} = A \cap B$, l.u.b.$\{A, B\}$ is the subspace $A + B$ spanned by A and B.

We are used to denote $a \wedge b$ for g.l.b.$\{a, b\}$ and $a \vee b$ for l.u.b.$\{a, b\}$. If $(L; \leqslant)$ is a lower semilattice, then $\wedge$ is a binary operation on L and we can induce an algebra $(L; \wedge)$ of type 2, satisfying the following conditions:

(1) idempotent law: $a \wedge a = a$;
(2) commutative law: $a \wedge b = b \wedge a$;
(3) associative law: $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

The converse is still true. That is because we can induce the following partial ordering $\leqslant$ on L such that $(L; \leqslant)$ is a lower semilattice:

$$a \leqslant b \text{ if and only if } a \wedge b = a \text{ for all } a,\ b \in L.$$

For the case that $(L; \leqslant)$ is an upper semilattice, there is also a similar situation. Then, as we have known, we have an alternative definition of lattices as follows. An algebra $(L; \wedge, \vee)$ of type (2, 2) is called a **lattice** if the following laws hold:

(L_1) idempotent law: $a \wedge a = a$ and $a \vee a = a$;
(L_2) commutative law: $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$;
(L_3) associative law: $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$;
(L_4) absorptive law: $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$.

From our definition of subalgebras, a **sublattice** M of a lattice $(L; \wedge, \vee)$ means that $M \neq \varnothing$ and M is closed under $\wedge$ and $\vee$ (here, the laws L_1 to L_4 are naturally valid in M). Then M with respect to the induced partial ordering $\leqslant$ forms a lattice $(M; \leqslant)$, where $a \leqslant b$ if and only if $a \wedge b = a$ (or equivalently, $a \vee b = b$). It is worth attending that given a nonempty subset of a lattice $(L; \leqslant)$, it with respect to $\leqslant$ may form a lattice where $\leqslant$ is the partial ordering defined on L, but such a lattice may not be a sublattice of $(L; \leqslant)$. For instance, the subspace lattice $(L(V); \subseteq)$ of a vector space V is generally not a sublattice of the power set lattice $(2^V; \subseteq)$ of V because the union $A \cup B$ of two subspaces A and B need not be a subspace of V. The occurrence of this phenomenon results from which the partial ordering $\leqslant$ is not a binary operation on L.

A lattice L is called **modular** if it satisfies the modular law:

$$a \geqslant b \text{ implies } a \wedge (b \vee c) = b \vee (a \wedge c),$$

or equivalently

$$a \leqslant b \text{ implies } a \vee (b \wedge c) = b \wedge (a \vee c).$$

All of the lattices in Examples 0.2.1 and 0.2.2 are modular except the pentagon lattice. From lattice theory, a lattice L is modular if and only if it does not contain any pentagon sublattices of L.

A totally ordered subset of a partially ordered set L is called a **chain**. An element a in L is said a **cover** of another element b in L if $a > b$ and there does not exist any element x in L such that $a > x > b$. A **connected chain** from a to b is a chain

$$a = a_0 > a_1 > a_2 > \cdots > a_n = b$$

such that a_{i-1} covers a_i, $i = 1, 2, \cdots, n$. In this case the number n is called the **length** of this chain. The greatest number in the lengths of all connected chains from a to b is said the **length** from a to b. If there is not such a greatest number, we say the length from a to b is **infinite**. If L contains the zero element 0, the length from a to 0 is often called the **length** of a. A partially ordered set is said to be of **finite length** if the lengths of all connected chains are bounded. The following is an interesting result in lattice theory.

Theorem 0.2.1. *Let a, b be elements in a modular lattice L such that $a > b$. If L is of finite length, then all connected chains from a to b have the same length.*

A lattice L is called **distributive** if it satisfies the distributive law:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$

or equivalently

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Every totally ordered set is obviously a distributive lattice. In Examples 0.2.1 and 0.2.2, the power set lattice is the only distributive lattice. As is well known, a distributive lattice must be modular, but the inverse is false. It is worth pointing out that from lattice theory, a distributive lattice is in essence a set algebra because it is isomorphic to a sublattice of the power set lattice 2^S of some set S. The following is a useful criterion for the distributivity of a lattice.

Theorem 0.2.2. *A lattice L is distributive if and only if it contains neither a pentagon sublattice nor a rhombus sublattice.*

Let L be a lattice with the zero element 0 and the unit element 1. Given a pair of elements a, b in L, if $a \wedge b = 0$ and $a \vee b = 1$, then one of a and b is called a **complement** of the other. If every $a \in L$ has its complements, we say L is a **complemented lattice**. The rhombus and pentagon lattices are complemented, but not distributive. A totally ordered set is a distributive lattice, but not complemented if the order of it is greater than 2. Generally speaking, the complements of an element are not unique if they exist. For instance, for the rhombus lattice in Example 0.2.1, y and z are the complements of x. However, as we have known, for a distributive lattice L with 0 and 1, the complement of an element a in L must be unique if it exists. We denote a' for the only complement of a.

If a lattice is both complemented and distributive, we call it a **Boolean algebra**, or a **Boolean lattice**. The symbol B is used to denote such a lattice. As any element a in B has one and only one complement a', there is a unary operation $'$ on B. Consequently, a Boolean algebra B is actually an algebra $(B; \wedge, \vee, ', 0, 1)$ of type (2, 2, 1, 0, 0). Every power set lattice 2^S is of course Boolean. Note that a distributive lattice is a set algebra in the sense of isomorphisms. A Boolean algebra is in reality a subalgebra of the algebra $(2^S; \cap, \cup, ', \varnothing, S)$ for some set S, where A' is the complementary set of A, i.e., $A' = S - A$ for any $A \in 2^S$. From this, the following laws are always true in a Boolean algebra:

(1) involution law: $a'' = a$ where $a'' = (a')'$;

(2) de Morgan's law: $(a \wedge b)' = a' \vee b'$ and $(a \vee b)' = a' \wedge b'$.

It has been known that a **ring** $(R; +, \cdot, 0)$ means that $(R; +, 0)$ is an Abelian group and $(R; \cdot)$ is a semigroup (i.e., R is closed under the multiplication and the associative law of multiplication holds) such that the left and right distributive laws of the multiplication to the addition are valid.

A **Boolean ring** $(B; +, \cdot, 0, 1)$ is a ring with 1 as the unit element such that each element $a \in B$ is idempotent (i.e., $a^2 = a$). For a Boolean ring B we have the following facts:

(1) B is of characteristic 2: $a + a = 0$ for all $a \in B$;

(2) the multiplication satisfies the commutative law: $ab = ba$;

(3) every element in $B - \{0, 1\}$ is a zero divisor: for any $a \in B - \{0, 1\}$, there is a nonzero element $b \in B$ (e.g., $b = 1 + a$) such that $ab = 0$.

Let $(B; \wedge, \vee, ', 0, 1)$ be a Boolean algebra. Define two binary operations $+$ and $\cdot$ on B by

$$a + b = (a \wedge b') \vee (a' \wedge b) \text{ and } a \cdot b = a \wedge b.$$

Then $(B; +, \cdot, 0, 1)$ is a Boolean ring (here, the verification is routine and omitted, the same below). For this ring, letting

$$a \sqcap b = ab, \ a \sqcup b = a + b + ab \text{ and } a^* = 1 + a,$$

we also have a Boolean algebra $(B; \sqcap, \sqcup, {}^*, 0, 1)$. It is interesting that we have the following facts:

$$a \sqcap b = a \wedge b, \ a \sqcup b = a \vee b \text{ and } a^* = a',$$

in other words, $(B; \sqcap, \sqcup, {}^*, 0, 1)$ is just the original algebra. Next, if we begin with a Boolean ring $(B; +, \cdot, 0, 1)$, we can induce a Boolean algebra $(B; \wedge, \vee, ', 0, 1)$ where

$$a \wedge b = ab, \ a \vee b = a + b + ab \text{ and } a' = 1 + a.$$

And then we can also induce a Boolean ring $(B; \oplus, \odot, 0, 1)$ where

$$a \oplus b = (a \wedge b') \vee (a' \wedge b) \text{ and } a \odot b = a \wedge b.$$

It is also interesting that $(B; \oplus, \odot, 0, 1)$ is just the original ring. These analyses show that the process of passing from a Boolean algebra to a Boolean ring and the process of passing from a Boolean ring to a Boolean algebra are inverses. We state these phenomena as the following theorem.

Theorem 0.2.3. *Boolean algebra and Boolean ring are two types of equivalent abstract systems.*

Finally, we state several terminologies as follows. Let L be a lattice. An **ideal** I of L means that I is a nonempty subset of L, satisfying the following conditions: for any $a, b, c \in L$,

(1) $a \in I$ and $b \in I$ imply $a \vee b \in I$;
(2) $a \in I$ and $c \leqslant a$ imply $c \in I$.

Dually, a **filter** or a **dual ideal** F of L is a nonempty subset of L, satisfying

(1) $a \in F$ and $b \in F$ imply $a \wedge b \in F$;
(2) $a \in F$ and $c \geqslant a$ imply $c \in F$.

Given an element $u \in L$, the set $(u] = \{a \in L \mid a \leqslant u\}$ is an ideal of L. Dually, the set $[u) = \{a \in L \mid a \geqslant u\}$ is a filter of L. It is easy to see that an ideal I of L is a sublattice of L, so is a filter F of L.

A mapping f from a lattice $(L; \wedge, \vee)$ to another lattice $(L'; \wedge', \vee')$ is called a **homomorphism** if for all $a, b \in L$,

(1) $f(a \wedge b) = f(a) \wedge' f(b)$;
(2) $f(a \vee b) = f(a) \vee' f(b)$.

Every lattice homomorphism $f : L \to L'$ is isotonic in the sense that

$$a \leqslant b \text{ implies } f(a) \leqslant' f(b) \text{ for any } a, b \in L.$$