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David Gilbarg · Neil S. Trudinger

Elliptic Partial Differential Equations of Second Order

二阶椭圆偏微分方程

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Preface to the Revised Third Printing

This revision of the 1983 second edition of "Elliptic Partial Differential Equations of Second Order" corresponds to the Russian edition, published in 1989, in which we essentially updated the previous version to 1984. The additional text relates to the boundary Hölder derivative estimates of Nikolai Krylov, which provided a fundamental component of the further development of the classical theory of elliptic (and parabolic), fully nonlinear equations in higher dimensions. In our presentation we adapted a simplification of Krylov's approach due to Luis Caffarelli.

The theory of nonlinear second order elliptic equations has continued to flourish during the last fifteen years and, in a brief epilogue to this volume, we signal some of the major advances. Although a proper treatment would necessitate at least another monograph, it is our hope that this book, most of whose text is now more than twenty years old, can continue to serve as background for these and future developments.

Since our first edition we have become indebted to numerous colleagues, all over the globe. It was particularly pleasant in recent years to make and renew friendships with our Russian colleagues, Olga Ladyzhenskaya, Nina Ural'tseva, Nina Ivochkina, Nikolai Krylov and Mikhail Safonov, who have contributed so much to this area. Sadly, we mourn the passing away in 1996 of Ennico De Giorgi, whose brilliant discovery forty years ago opened the door to the higher-dimensional nonlinear theory.

October 1997 *David Gilbarg · Neil S. Trudinger*

Preface to the First Edition

This volume is intended as an essentially self-contained exposition of portions of the theory of second order quasilinear elliptic partial differential equations, with emphasis on the Dirichlet problem in bounded domains. It grew out of lecture notes for graduate courses by the authors at Stanford University, the final material extending well beyond the scope of these courses. By including preparatory chapters on topics such as potential theory and functional analysis, we have attempted to make the work accessible to a broad spectrum of readers. Above all, we hope the readers of this book will gain an appreciation of the multitude of ingenious barehanded techniques that have been developed in the study of elliptic equations and have become part of the repertoire of analysis.

Many individuals have assisted us during the evolution of this work over the past several years. In particular, we are grateful for the valuable discussions with L. M. Simon and his contributions in Sections 15.4 to 15.8; for the helpful comments and corrections of J. M. Cross, A. S. Geue, J. Nash, P. Trudinger and B. Turkington; for the contributions of G. Williams in Section 10.5 and of A. S. Geue in Section 10.6; and for the impeccably typed manuscript which resulted from the dedicated efforts of Isolde Field at Stanford and Anna Zalucki at Canberra. The research of the authors connected with this volume was supported in part by the National Science Foundation.

August 1977

David Gilbarg
Stanford

Neil S. Trudinger
Canberra

Note: The Second Edition includes a new, additional Chapter 9. Consequently Chapters 10 and 15 referred to above have become Chapters 11 and 16.

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Part II. Quasilinear Equations

Chapter 1

Introduction

Summary

The principal objective of this work is the systematic development of the general theory of second order quasilinear elliptic equations and of the linear theory required in the process. This means we shall be concerned with the solvability of boundary value problems (primarily the Dirichlet problem) and related general properties of solutions of linear equations

$$(1.1) \qquad Lu \equiv a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u = f(x), \quad i, j = 1, 2, \ldots, n,$$

and of quasilinear equations

$$(1.2) \qquad Qu \equiv a^{ij}(x, u, Du)D_{ij}u + b(x, u, Du) = 0.$$

Here $Du = (D_1u, \ldots, D_nu)$, where $D_iu = \partial u/\partial x_i$, $D_{ij}u = \partial^2 u/\partial x_i\, \partial x_j$, etc., and the summation convention is understood. The ellipticity of these equations is expressed by the fact that the coefficient matrix $[a^{ij}]$ is (in each case) positive definite in the domain of the respective arguments. We refer to an equation as *uniformly elliptic* if the ratio γ of maximum to minimum eigenvalue of the matrix $[a^{ij}]$ is bounded. We shall be concerned with both non-uniformly and uniformly elliptic equations.

The classical prototypes of linear elliptic equations are of course Laplace's equation

$$\Delta u = \sum D_{ii}u = 0$$

and its inhomogeneous counterpart, Poisson's equation $\Delta u = f$. Probably the best known example of a quasilinear elliptic equation is the minimal surface equation

$$\sum D_i(D_iu/(1 + |Du|^2)^{1/2}) = 0,$$

which arises in the problem of least area. This equation is non-uniformly elliptic, with $\gamma = 1 + |Du|^2$. The properties of the differential operators in these examples motivate much of the theory of the general classes of equations discussed in this book.

The relevant linear theory is developed in Chapters 2–9 (and in part of Chapter 12). Although this material has independent interest, the emphasis here is on aspects needed for application to nonlinear problems. Thus the theory stresses weak hypotheses on the coefficients and passes over many of the important classical and modern results on linear elliptic equations.

Since we are ultimately interested in classical solutions of equation (1.2), what is required at some point is an underlying theory of classical solutions for a sufficiently large class of linear equations. This is provided by the Schauder theory in Chapter 6, which is an essentially complete theory for the class of equations (1.1) with Hölder continuous coefficients. Whereas such equations enjoy a definitive existence and regularity theory for classical solutions, corresponding results cease to be valid for equations in which the coefficients are assumed only continuous.

A natural starting point for the study of classical solutions is the theory of Laplace's and Poisson's equations. This is the content of Chapters 2 and 4. In anticipation of later developments the Dirichlet problem for harmonic functions with continuous boundary values is approached through the Perron method of subharmonic functions. This emphasizes the maximum principle, and with it the barrier concept for studying boundary behavior, in arguments that are readily extended to more general situations in later chapters. In Chapter 4 we derive the basic Hölder estimates for Poisson's equation from an analysis of the Newtonian potential. The principal result here (see Theorems 4.6, 4.8) states that all $C^2(\Omega)$ solutions of Poisson's equation, $\Delta u=f$, in a domain Ω of $\mathbb{R}^n$ satisfy a uniform estimate in any subset $\Omega'\subset\subset\Omega$

$$(1.3)\qquad \|u\|_{C^{2,\alpha}(\bar{\Omega}')}\leqslant C(\sup_{\Omega}|u|+\|f\|_{C^{\alpha}(\bar{\Omega})}),$$

where C is a constant depending only on α $(0<\alpha<1)$, the dimension n and dist $(\Omega', \partial\Omega)$; (for notation see Section 4.1). This *interior estimate* (interior since $\Omega'\subset\subset\Omega$) can be extended to a *global estimate* for solutions with sufficiently smooth boundary values provided the boundary $\partial\Omega$ is also sufficiently smooth. In Chapter 4 estimates up to the boundary are established only for hyperplane and spherical boundaries, but these suffice for the later applications.

The climax of the theory of classical solutions of linear second order elliptic equations is achieved in the Schauder theory, which is developed in modified and expanded form in Chapter 6. Essentially, this theory extends the results of potential theory to the class of equations (1.1) having Hölder continuous coefficients. This is accomplished by the simple but fundamental device of regarding the equation locally as a perturbation of the constant coefficient equation obtained by fixing the leading coefficients at their values at a single point. A careful calculation based on the above mentioned estimates for Poisson's equation yields the same inequality (1.3) for any $C^{2,\alpha}$ solution of (1.1), where the constant C now depends also on the bounds and Hölder constants of the coefficients and in addition on the minimum and maximum eigenvalues of the coefficient matrix $[a^{ij}]$ in Ω. These results are stated as interior estimates in terms of weighted interior norms (Theorem 6.2) and, in the case of sufficiently smooth boundary data, as global estimates in terms of global norms (Theorem 6.6). Here we meet the important and recurring concept of an *apriori* estimate; namely, an estimate (in terms of given data) valid for all *possible* solutions of a class of problems even if the hypotheses do not guarantee the existence of such solutions. A major part of this book is devoted to the establishment of apriori bounds for various problems. (We have taken the liberty of replacing the latin *a priori* with the single word *apriori*, which will be used throughout.)

The importance of such apriori estimates is visible in several applications in Chapter 6, among them in establishing the solvability of the Dirichlet problem by the method of continuity (Theorem 6.8) and in proving the higher order regularity of C^2 solutions under appropriate smoothness hypotheses (Theorems 6.17, 6.19). In both cases the estimates provide the necessary compactness properties for certain classes of solutions, from which the desired results are easily inferred.

We remark on several additional features of Chapter 6, which are not needed for the later developments but which broaden the scope of the basic Schauder theory. In Section 6.5 it is seen that for continuous boundary values and a suitably wide class of domains the proof of solvability of the Dirichlet problem for (1.1) can be achieved entirely with interior estimates, thereby simplifying the structure of the theory. The results of Section 6.6 extend the existence theory for the Dirichlet problem to certain classes of non-uniformly elliptic equations. Here we see how relations between geometric properties of the boundary and the degenerate ellipticity at the boundary determine the continuous assumption of boundary values. The methods are based on barrier arguments that foreshadow analogous (but deeper) results for nonlinear equations in Part II. In Section 6.7 we extend the theory of (1.1) to the regular oblique derivative problem. The method is basically an extrapolation to these boundary conditions of the earlier treatment of Poisson's equation and the Schauder theory (without barrier arguments, however).

In the preceding considerations, especially in the existence theory and barrier arguments, the maximum principle for the operator L (when $c \leqslant 0$) plays an essential part. This is a special feature of second order elliptic equations that simplifies and strengthens the theory. The basic facts concerning the maximum principle, as well as illustrative applications of comparison methods, are contained in Chapter 3. The maximum principle provides the earliest and simplest apriori estimates of the general theory. It is of considerable interest that all the estimates of Chapters 4 and 6 can be derived entirely from comparison arguments based on the maximum principle, without any mention of the Newtonian potential or integrals.

An alternative and more general approach to linear problems, without potential theory, can be achieved by Hilbert space methods based on *generalized* or *weak* solutions, as in Chapter 8. To be more specific, let L' be a second order differential operator, with principal part of *divergence form*, defined by

$$L'u \equiv D_i(a^{ij}(x)D_ju + b^i(x)u) + c^i(x)D_iu + d(x)u.$$

If the coefficients are sufficiently smooth, then clearly this operator falls within the class discussed in Chapter 6. However, even if the coefficients are in a much wider class and u is only weakly differentiable (in the sense of Chapter 7), one can still define weak or generalized solutions of $L'u=g$ in appropriate function classes. In particular, if the coefficients a^{ij}, b^i, c^i are bounded and measurable in Ω and g is an integrable function in Ω, let us call u a weak or generalized solution of $L'u=g$ in Ω if $u \in W^{1,2}(\Omega)$ (as defined in Chapter 7) and

$$(1.4) \qquad \int_\Omega [(a^{ij}D_ju+b^iu)D_iv-(c^iD_iu+du)v]\,dx=-\int_\Omega gv\,dx$$

for all *test functions* $v \in C_0^1(\Omega)$. It is clear that if the coefficients and g are sufficiently smooth and $u \in C^2(\Omega)$, then u is also a classical solution.

We can now speak also of weak solutions u of the *generalized Dirichlet problem*,

$$L'u=g \text{ in } \Omega, \quad u=\varphi \text{ on } \partial\Omega,$$

if u is a weak solution satisfying $u-\varphi \in W_0^{1,2}(\Omega)$, where $\varphi \in W^{1,2}(\Omega)$. Assuming that the minimum eigenvalue of $[a^{ij}]$ is bounded away from zero in Ω, that

$$(1.5) \qquad D_ib^i+d\leqslant 0$$

in the weak sense, and that also $g \in L^2(\Omega)$, we find in Theorem 8.3 that the generalized Dirichlet problem has a unique solution $u \in W^{1,2}(\Omega)$. Condition (1.5), which is the analogue of $c\leqslant 0$ in (1.1), assures a maximum principle for weak solutions of $L'u\geqslant 0(\leqslant 0)$ (Theorem 8.1) and hence uniqueness for the generalized Dirichlet problem. Existence of a solution then follows from the Fredholm alternative for the operator L' (Theorem 8.6), which is proved by an application of the Riesz representation theorem in the Hilbert space $W_0^{1,2}(\Omega)$.

The major part of Chapter 8 is taken up with the regularity theory for weak solutions. Additional regularity of the coefficients in (1.4) implies that the solutions belong to higher $W^{k,2}$ spaces (Theorems 8.8, 8.10). It follows from the Sobolev imbedding theorems in Chapter 7 that weak solutions are in fact classical solutions provided the coefficients are sufficiently regular. Global regularity of these solutions is inferred by extending interior regularity to the boundary when the boundary data are sufficiently smooth (Theorems 8.13, 8.14).

The regularity theory of weak solutions and the associated pointwise estimates are fundamental to the nonlinear theory. These results provide the starting point for the "bootstrap" arguments that are typical of nonlinear problems. Briefly, the idea here is to start with weak solutions of a quasilinear equation, regarding them as weak solutions of related *linear* equations obtained by inserting them into the coefficients, and then to proceed by establishing improved regularity of these solutions. Starting anew with the latter solutions and repeating the process, still further regularity is assured, and so on, until the original weak solutions are finally proved to be suitably smooth. This is the essence of the regularity proofs for the older variational problems and is implicit in the nonlinear theory presented here.

The Hölder estimates for weak solutions that are so vital for the nonlinear theory are derived in Chapter 8 from Harnack inequalities based on the Moser iteration technique (Theorems 8.17, 8.18, 8.20, 8.24). These results generalize the basic apriori Hölder estimate of De Giorgi, which provided the initial breakthrough in the theory of quasilinear equations in more than two independent variables. The arguments rest on integral estimates for weak solutions u derived from judicious choice of test functions v in (1.4). The test function technique is the dominant theme in the derivation of estimates throughout most of this work.

In this edition we have added new material to Chapter 8 covering the Wiener criterion for regular boundary points, eigenvalues and eigenfunctions, and Hölder estimates for first derivatives of solutions of linear divergence structure equations.

We conclude Part I of the present edition with a new chapter, Chapter 9, concerning strong solutions of linear elliptic equations. These are solutions which possess second derivatives, at least in a weak sense, and satisfy (1.1) almost everywhere. Two strands are interwoven in this chapter. First we derive a maximum principle of Aleksandrov, and a related apriori bound (Theorem 9.1) for solutions in the Sobolev space $W^{2,n}(\Omega)$, thereby extending certain basic results from Chapter 3 to nonclassical solutions. Later in the chapter, these results are applied to establish various pointwise estimates, including the recent Hölder and Harnack estimates of Krylov and Safonov (Theorems 9.20, 9.22; Corollaries 9.24, 9.25). The other strand in this chapter is the L^p theory of linear second-order elliptic equations that is analogous to the Schauder theory of Chapter 6. The basic estimate for Poisson's equation, namely the Calderon-Zygmund inequality (Theorem 9.9) is derived through the Marcinkiewicz interpolation theorem, although without the use of Fourier transform methods. Interior and global estimates in the Sobolev spaces $W^{2,p}(\Omega)$, $1 < p < \infty$, are established in Theorems 9.11, 9.13 and applied to the Dirichlet problem for strong solutions, in Theorem 9.15 and Corollary 9.18.

Part II of this book is devoted largely to the Dirichlet problem and related estimates for quasilinear equations. The results concern in part the general operator (1.2) while others apply especially to operators of divergence form

$$(1.6) \qquad Qu \equiv \operatorname{div} \mathbf{A}(x, u, Du) + B(x, u, Du)$$

where $\mathbf{A}(x, z, p)$ and $B(x, z, p)$ are respectively vector and scalar functions defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$.

Chapter 10 extends maximum and comparison principles (analogous to results in Chapter 3) to solutions and subsolutions of quasilinear equations. We mention in particular apriori bounds for solutions of $Qu \geqslant 0$ $(=0)$, where Q is a divergence form operator satisfying certain structure conditions more general than ellipticity (Theorem 10.9).

Chapter 11 provides the basic framework for the solution of the Dirichlet problem in the following chapters. We are concerned principally with classical solutions, and the equations may be uniformly or non-uniformly elliptic. Under suitable general hypotheses any globally smooth solution u of the boundary value problem for $Qu=0$ in a domain Ω with smooth boundary can be viewed as a fixed point, $u = Tu$, of a compact operator T from $C^{1,\alpha}(\bar{\Omega})$ to $C^{1,\alpha}(\bar{\Omega})$ for any $\alpha \in (0,1)$. In the applications the function defined by Tu, for any $u \in C^{1,\alpha}(\bar{\Omega})$, is the unique solution of the *linear* problem obtained by inserting u into the coefficients of Q. The Leray-Schauder fixed point theorem (proved in Chapter 11) then implies the existence of a solution of the boundary value problem provided an apriori bound, in $C^{1,\alpha}(\bar{\Omega})$, can be established for the solutions of a related continuous family of equations $u = T(u;\sigma)$, $0 \leqslant \sigma \leqslant 1$, where $T(u;1) = Tu$ (Theorems 11.4, 11.6). The establishment of such bounds for certain broad classes of Dirichlet problems is the object of Chapters 13–15.

The general procedure for obtaining the required apriori bound for possible solutions u is a four-step process involving successive estimation of $\sup_\Omega |u|$, $\sup_{\partial\Omega} |Du|$, $\sup_\Omega |Du|$, and $\|u\|_{C^{1,\alpha}(\bar{\Omega})}$ for some $\alpha > 0$. Each of these estimates presupposes the preceding ones and the final bound on $\|u\|_{C^{1,\alpha}(\bar{\Omega})}$ completes the existence proof based on the Leray-Schauder theorem.

As already observed, bounds on $\sup_\Omega |u|$ are discussed in Chapter 10. In the later chapters this bound is either assumed in the hypotheses or is implied by properties of the equation.

Equations in two variables (Chapter 12) occupy a special place in the theory. This is due in part to the distinctive methods that have been developed for them and also to the results, some of which have no counterpart for equations in more than two variables. The method of quasiconformal mappings and arguments based on divergence structure equations (cf. Chapter 11) are both applicable to equations in two variables and yield relatively easily the desired $C^{1,\alpha}$ apriori estimates, from which a solution of the Dirichlet problem follows readily.

Of particular interest is the fact that solutions of uniformly elliptic linear equations in two variables satisfy an apriori $C^{1,\alpha}$ estimate depending only on the ellipticity constants and bounds on the coefficients, without any regularity assumptions (Theorem 12.4). Such a $C^{1,\alpha}$ estimate, or even the existence of a gradient bound under the same general conditions is unknown for equations in more than two variables. Another special feature of the two-dimensional theory is the existence of an apriori C^1 bound $|Du| \leqslant K$ for solutions of *arbitrary* elliptic equations

$$au_{xx} + 2bu_{xy} + cu_{yy} = 0, \tag{1.7}$$

where u is continuous on the closure of a bounded convex domain Ω and has boundary value φ on $\partial\Omega$ satisfying a bounded slope (or three-point) condition with constant K. This classical result, usually based on a theorem of Radó on saddle surfaces, is given an elementary proof in Lemma 12.6. The stated gradient bound, which is valid for all solutions u of the general quasilinear equation (1.7) in which $a = a(x, y, u, u_x, u_y)$, etc., and such that $u = \varphi$ on $\partial\Omega$, reduces this Dirichlet problem to the case of uniformly elliptic equations treated in Theorem 12.5. In Theorem 12.7 we obtain a solution of the general Dirichlet problem for (1.7), assuming local Hölder continuity of the coefficients and a bounded slope condition for the boundary data (without further smoothness restrictions on the data).

Chapters 13, 14 and 15 are devoted to the derivation of the gradient estimates involved in the existence procedure described above. In Chapter 13, we prove the fundamental results of Ladyzhenskaya and Ural'tseva on Hölder estimates of derivatives of elliptic quasilinear equations. In Chapter 14 we study the estimation of the gradient of solutions of elliptic quasilinear equations on the boundary.