# Graduate Texts in Mathematics

P. J. Hilton
U. Stammbach

# A Course in Homological Algebra

## Second Edition

# 同调代数教程

## 第2版

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*To Margaret and Irene*

## Preface to the Second Edition

We have inserted, in this edition, an extra chapter (Chapter X) entitled "Some Applications and Recent Developments." The first section of this chapter describes how homological algebra arose by abstraction from algebraic topology and how it has contributed to the knowledge of topology. The other four sections describe applications of the methods and results of homological algebra to other parts of algebra. Most of the material presented in these four sections was not available when this text was first published. Naturally, the treatments in these five sections are somewhat cursory, the intention being to give the flavor of the homological methods rather than the details of the arguments and results.

We would like to express our appreciation of help received in writing Chapter X; in particular, to Ross Geoghegan and Peter Kropholler (Section 3), and to Jacques Thévenaz (Sections 4 and 5).

The only other changes consist of the correction of small errors and, of course, the enlargement of the Index.

Binghamton, New York, USA
Zürich, Switzerland

Peter Hilton
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# Contents

# Introduction*

This book arose out of a course of lectures given at the Swiss Federal Institute of Technology (ETH), Zürich, in 1966–67. The course was first set down as a set of lecture notes, and, in 1968, Professor Eckmann persuaded the authors to build a graduate text out of the notes, taking account, where appropriate, of recent developments in the subject.

The level and duration of the original course corresponded essentially to that of a year-long, first-year graduate course at an American university. The background assumed of the student consisted of little more than the algebraic theories of finitely-generated abelian groups and of vector spaces over a field. In particular, he was not supposed to have had any formal instruction in categorical notions beyond simply some understanding of the basic terms employed (category, functor, natural transformation). On the other hand, the student was expected to have some sophistication and some preparation for rather abstract ideas. Further, no knowledge of algebraic topology was assumed, so that such notions as chain-complex, chain-map, chain-homotopy, homology were not already available and had to be introduced as purely algebraic constructs. Although references to relevant ideas in algebraic topology do feature in this text, as they did in the course, they are in the nature of (two-way) motivational enrichment, and the student is not left to depend on any understanding of topology to provide a justification for presenting a given topic.

The level and knowledge assumed of the student explains the order of events in the opening chapters. Thus, Chapter I is devoted to the theory of modules over a unitary ring $\Lambda$. In this chapter, we do little more than introduce the category of modules and the basic functors on modules and the notions of projective and injective modules, together with their most easily accessible properties. However, on completion of Chapter I, the student is ready with a set of examples to illumine his understanding of the abstract notions of category theory which are presented in Chapter II.

* Sections of this Introduction in small type are intended to give amplified motivation and background for the more experienced algebraist. They may be ignored, at least on first reading, by the beginning graduate student.

In this chapter we are largely influenced in our choice of material by the demands of the rest of the book. However, we take the view that this is an opportunity for the student to grasp basic categorical notions which permeate so much of mathematics today, including, of course, algebraic topology, so that we do not allow ourselves to be rigidly restricted by our immediate objectives. A reader totally unfamiliar with category theory may find it easiest to restrict his first reading of Chapter II to Sections 1 to 6; large parts of the book are understandable with the material presented in these sections. Another reader, who had already met many examples of categorical formulations and concepts might, in fact, prefer to look at Chapter II before reading Chapter I. Of course the reader thoroughly familiar with category theory could, in principal, omit Chapter II, except perhaps to familiarize himself with the notations employed.

In Chapter III we begin the proper study of homological algebra by looking in particular at the group $\mathrm{Ext}_\Lambda(A, B)$, where $A$ and $B$ are $\Lambda$-modules. It is shown how this group can be calculated by means of a projective presentation of $A$, or an injective presentation of $B$; and how it may also be identified with the group of equivalence classes of extensions of the quotient module $A$ by the submodule $B$. These facets of the Ext functor are prototypes for the more general theorems to be presented later in the book. Exact sequences are obtained connecting Ext and Hom, again preparing the way for the more general results of Chapter IV. In the final sections of Chapter III, attention is turned from the Ext functor to the Tor functor, $\mathrm{Tor}^\Lambda(A, B)$, which is related to the tensor product of a right $\Lambda$-module $A$ and a left $\Lambda$-module $B$ rather in the same way as Ext is related to Hom.

With the special cases of Chapter III mastered, the reader should be ready at the outset of Chapter IV for the general idea of a derived functor of an additive functor which we regard as the main motif of homological algebra. Thus, one may say that the material prior to Chapter IV constitutes a build-up, in terms of mathematical knowledge and the study of special cases, for the central ideas of homological algebra which are presented in Chapter IV. We introduce, quite explicitly, left and right derived functors of both covariant and contravariant additive functors, and we draw attention to the special cases of right-exact and left-exact functors. We obtain the basic exact sequences and prove the *balance* of $\mathrm{Ext}^n_\Lambda(A, B)$, $\mathrm{Tor}^\Lambda_n(A, B)$ as bifunctors. It would be reasonable to regard the first four chapters as constituting the first part of the book, as they did, in fact, of the course.

Chapter V is concerned with a very special situation of great importance in algebraic topology where we are concerned with tensor products of free abelian chain-complexes. There it is known that there is a formula expressing the homology groups of the tensor product of the free abelian chain-complexes $\boldsymbol{C}$ and $\boldsymbol{D}$ in terms of the homology groups of $\boldsymbol{C}$ and $\boldsymbol{D}$. We generalize this Künneth formula and we also give a corresponding formula in which the tensor product is replaced by Hom. This corresponding formula is not of such immediate application to topology (where the Künneth formula for the tensor product yields a significant result in the homology of topological products), but it is valuable in homological algebra and leads to certain important identities relating Hom, Ext, tensor and Tor.

Chapters VI and VII may, in a sense, be regarded as individual monographs. In Chapter VI we discuss the homology theory of abstract groups. This is the most classical topic in homological algebra and really provided the original impetus for the entire development of the subject. It has seemed to us important to go in some detail into this theory in order to provide strong motivation for the abstract ideas introduced. Thus, we have been concerned in particular to show how homological ideas may yield proofs of results in group theory which do not require any homology theory for their formulation – and indeed, which were enunciated and proved in some cases before or without the use of homological ideas. Such an example is Maschke's theorem which we state and prove in Section 16.

The relation of the homology theory of groups to algebraic topology is explained in the introductory remarks in Chapter VI itself. It would perhaps be appropriate here to give some indication of the scope and application of the homology theory of groups in group theory. Eilenberg and MacLane [15] showed that the second cohomology group, $H^2(G, A)$, of the group $G$ with coefficients in the $G$-module $A$, may be used to formalize the extension theory of groups due to Schreier, Baer, and Fitting. They also gave an interpretation of $H^3(G,A)$ in terms of group extensions with non-abelian kernel, in which $A$ plays the role of the center of the kernel. For a contemporary account of these theories, see Gruenberg [20]. In subsequent developments, the theory has been applied extensively to finite groups and to class field theory by Hochschild, Tate, Artin, etc.; see Weiss [49]. A separate branch of cohomology, the so-called Galois cohomology, has grown out of this connection and has been extensively studied by many algebraists (see Serre [41]).

The natural ring structure in the cohomology of groups, which is clearly in evidence in the relation of the cohomology of a group to that of a space, has also been studied, though not so extensively. However, we should mention here the deep result of L. Evens [17] that the cohomology ring of a finite group is finitely generated.

It would also be appropriate to mention the connection which has been established between the homology theory of groups and algebraic $K$-theory, a very active area of mathematical research today, which seems to offer hope of providing us with an effective set of invariants of unitary rings. Given a unitary ring $\Lambda$ we may form the general linear group, $GL_n(\Lambda)$, of invertible $(n \times n)$ matrices over $\Lambda$, and then the group $GL(\Lambda)$ is defined to be the union of the groups $GL_n(\Lambda)$ under the natural inclusions. If $E(\Lambda)$ is the commutator subgroup of $GL(\Lambda)$, then a definition given by Milnor for $K_2(\Lambda)$, in terms of the Steinberg group, amounts to saying that $K_2(\Lambda) = H_2(E(\Lambda))$. Moreover, the group $E(\Lambda)$ is perfect, that is to say, $H_1(E(\Lambda)) = 0$, so that the study of the $K$-groups of $\Lambda$ leads to the study of the second homology group of perfect groups. The second homology group of the group $G$ actually has an extremely long history, being effectively the *Schur multiplicator* of $G$, as introduced by Schur [40] in 1904.

Finally, to indicate the extent of activity in this area of algebra, without in any way trying to be comprehensive, we should refer to the proof by Stallings [45] and Swan [48], that a group $G$ is free if and only if $H^n(G, A) = 0$ for all $G$-modules $A$ and all $n \geqq 2$. That the cohomology vanishes in dimensions $\geqq 2$ when $G$ is free is quite trivial (and is, of course, proved in this book); the opposite implication, however, is deep and difficult to establish. The result has particularly interesting consequences for torsion-free groups.

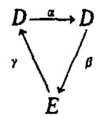
In Chapter VII we discuss the cohomology theory of Lie algebras. Here the spirit and treatment are very much the same as in Chapter VI, but we do not treat Lie algebras so extensively, principally because so much of the development is formally analogous to that for the cohomology of groups. As explained in the introductory remarks to the chapter, the cohomology theory of Lie algebras, like the homology theory of groups, arose originally from considerations of algebraic topology, namely, the cohomology of the underlying spaces of Lie groups. However, the theory of Lie algebra cohomology has developed independently of its topological origins.

This development has been largely due to the work of Koszul [31]. The cohomological proofs of two main theorems of Lie algebra theory which we give in Sections 5 and 6 of Chapter VII are basically due to Chevalley-Eilenberg [8]. Hochschild [24] showed that, as for groups, the three-dimensional cohomology group $H^3(\mathfrak{g}, A)$ of the Lie algebra $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module $A$ classifies obstructions to extensions with non-abelian kernel.

Cartan and Eilenberg [7] realized that group cohomology and Lie algebra cohomology (as well as the cohomology of associative algebras over a field) may all be obtained by a general procedure, namely, as derived functors in a suitable module-category. It is, of course, this procedure which is adopted in this book, so that we have presented the theory of derived functors in Chapter IV as the core of homological algebra, and Chapters VI and VII are then treated as important special cases.

Chapters VIII and IX constitute the third part of the book. Chapter VIII consists of an extensive treatment of the theory of spectral sequences. Here, as in Chapter II, we have gone beyond the strict requirements of the applications which we make in the text. Since the theory of spectral sequences is so ubiquitous in homological algebra and its applications, it appeared to us to be sensible to give the reader a thorough grounding in the topic. However, we indicate in the introductory remarks to Chapter VIII, and in the course of the text itself, those parts of the chapter which may be omitted by the reader who simply wishes to be able to understand those applications which are explicitly presented. Our own treatment gives prominence to the idea of an exact couple and emphasizes the notion of the spectral sequence functor on the category of exact couples. This is by no means the unique way of presenting spectral sequences and the reader should, in particular, consult the book of Cartan-Eilenberg [7] to see an alternative approach. However, we do believe that the approach adopted is a reasonable one and a natural one. In fact, we have presented an elaboration of the notion of an exact couple, namely, that of a Rees system, since within the Rees system is contained all the information necessary to deduce the crucial convergence properties of the spectral sequence. Our treatment owes much to the study by Eckmann-Hilton [10] of exact couples in an abelian category. We take from them the point of view that the grading on the objects should only be introduced at such time as it is crucial for the study of convergence; that is to say, the purely algebraic constructions are carried out without any reference to grading. This, we believe, simplifies the presentation and facilitates the understanding.

We should point out that we depart in Chapter VIII from the standard conventions with regard to spectral sequences in one important and one less important respect. We index the original exact couple by the symbol 0 so that the first derived couple is indexed by the symbol 1 and, in general, the $n$th derived couple by the symbol $n$. This has the effect that what is called by most authorities the $E_2$-term appears with us as the $E_1$-term. We do not believe that this difference of convention, once it has been drawn to the attention of the reader, should cause any difficulties. On the other hand, we claim that the convention we adopt has many advantages. Principal among them, perhaps, is the fact that in the exact couple




the $n$th differential in the associated spectral sequence $d_n$ is, by our convention, induced by $\beta\alpha^{-n}\gamma$. With the more habitual convention $d_n$ would be induced by $\beta\alpha^{-n+1}\gamma$. It is our experience that where a difference of unity enters gratuitously into a formula like this, there is a great danger that the sign is misremembered – or that the difference is simply forgotten. A minor departure from the more usual convention is that the second index, or $q$ index, in the spectral sequence term, $E_r^{p,q}$, signifies the total degree and not the complementary degree. As a result, we have the situation that if $C$ is a filtered chain-complex, then $H_q(C)$ is filtered by subgroups whose associated graded group is $\{E_\infty^{p,q}\}$. Our convention is the one usually adopted for the generalized Atiyah-Hirzebruch spectral sequence, but it is not the one introduced by Serre in his seminal paper on the homology of fiber spaces, which has influenced the adoption of the alternative convention to which we referred above. However, since the translation from one convention to another is, in this case, absolutely trivial (with our convention, the term $E_r^{p,q}$ has complementary degree $q-p$), we do not think it necessary to lay further stress on this distinction.

Chapter IX is somewhat different from the other chapters in that it represents a further development of many of the ideas of the rest of the text, in particular, those of Chapters IV and VIII. This chapter did not appear in its present form in the course, which concluded with applications of spectral sequences available through the material already familiar to the students. In the text we have permitted ourselves further theoretical developments and generalizations. In particular, we present the theory of satellites, some relative homological algebra, and the theory of the homology of small categories. Since this chapter does constitute further development of the subject, one might regard its contents as more arbitrary than those of the other chapters and, in the same way, the chapter itself is far more open-ended than its predecessors. In particular, ideas are presented in the expectation that the student will be encouraged to make a further study of them beyond the scope of this book.

Each chapter is furnished with some introductory remarks describing the content of the chapter and providing some motivation and background. These introductory remarks are particularly extensive in the case of Chapters VI and VII in view of their special nature. The chapters are divided into sections and each section closes with a set of exercises.* These exercises are of many different kinds; some are purely computational, some are of a theoretical nature, and some ask the student to fill in gaps in the text where we have been content to omit proofs. Sometimes we suggest exercises which take the reader beyond the scope of the text. In some cases, exercises appearing at the end of a given section may reappear as text material in a later section or later chapter; in fact, the results stated in an exercise may even be quoted subsequently with appropriate reference, but this procedure is adopted only if their demonstration is incontestably elementary.

Although this text is primarily intended to accompany a course at the graduate level, we have also had in mind the obligation to write a book which can be used as a work of reference. Thus, we have endeavored, by giving very precise references, by making self-contained statements, and in other ways, to ensure that the reader interested in a particular aspect of the theory covered by the text may dip into the book at any point and find the material intelligible – always assuming, of course, that he is prepared to follow up the references given. This applies in particular to Chapters VI and VII, but the same principles have been adopted in designing the presentation in all the chapters.

The enumeration of items in the text follows the following conventions. The chapters are enumerated with Roman numerals and the

* Of course, Chapter X is different.

sections with Arabic numerals. Within a given chapter, we have two series of enumerations, one for theorems, lemmas, propositions, and corollaries, the other for displayed formulas. The system of enumeration in each of these series consists of a pair of numbers, the first referring to the section and the second to the particular item. Thus, in Section 5 of Chapter VI, we have Theorem 5.1 in which a formula is displayed which is labeled (5.2). On the subsequent page there appears Corollary 5.2 which is a corollary to Theorem 5.1. When we wish to refer to a theorem, etc., or a displayed formula, we simply use the same system of enumeration, provided the item to be cited occurs in the same chapter. If it occurs in a different chapter, we will then precede the pair of numbers specifying the item with the Roman numeral specifying the chapter. The exercises are enumerated according to the same principle. Thus, Exercise 1.2 of Chapter VIII refers to the second exercise at the end of the first section of Chapter VIII. A reference to Exercise 1.2, occurring in Chapter VIII, means Exercise 1.2 of that chapter. If we wish to refer to that exercise in the course of a different chapter, we would refer to Exercise VIII.1.2.

This text arose from a course and is designed, itself, to constitute a graduate course, at the first-year level at an American university. Thus, there is no attempt at complete coverage of all areas of homological algebra. This should explain the omission of such important topics as Hopf algebras, derived categories, triple cohomology, Galois cohomology, and others, from the content of the text. Since, in planning a course, it is necessary to be selective in choosing applications of the basic ideas of homological algebra, we simply claim that we have made one possible selection in the second and third parts of the text. We hope that the reader interested in applications of homological algebra not given in the text will be able to consult the appropriate authorities.

We have not provided a bibliography beyond a list of references to works cited in the text. The comprehensive listing by Steenrod of articles and books in homological algebra * should, we believe, serve as a more than adequate bibliography. Of course it is to be expected that the instructor in a course in homological algebra will, himself, draw the students' attention to further developments of the subject and will thus himself choose what further reading he wishes to advise. As a single exception to our intention not to provide an explicit bibliography, we should mention the work by Saunders MacLane, *Homology*, published by Springer-Verlag, which we would like to view as a companion volume to the present text.

Some remarks are in order about notational conventions. First, we use the left-handed convention, whereby the composite of the morphism $\varphi$

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* *Reviews of Papers in Algebraic and Differential Topology, Topological Groups and Homological Algebra*, Part II (American Mathematical Society).