CONTEMPORARY MATHEMATICS

610

Perspectives in Representation Theory

A Conference in Honor of Igor Frenkel's 60th Birthday
on Perspectives in Representation Theory
May 12–17, 2012
Yale University, New Haven, CT

Pavel Etingof
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2010 *Mathematics Subject Classification.* Primary 17Bxx, 22E57.

Library of Congress Cataloging-in-Publication Data

Perspectives in representation theory : a conference in honor of Igor Frenkel's 60th birthday : May 12–17, 2012, Yale University, New Haven, CT / Pavel Etingof, Mikhail Khovanov, Alistair Savage, editors.

pages cm. – (Contemporary mathematics ; volume 610)

Includes bibliographical references.

ISBN 978-0-8218-9170-4 (alk. paper)

1. Frenkel, Igor–Congresses. 2. Representations of algebras–Congresses. 3. Lie algebras–Congresses. 4. Representations of groups–Congresses. 5. Group theory–Congresses. I. Etingof, P. I. (Pavel I.), 1969- editor of compilation. II. Khovanov, Mikhail editor of compilation. III. Savage, Alistair editor of compilation.

QA176.P47 2014
512′.22–dc23   2013035921

Contemporary Mathematics ISSN: 0271-4132 (print); ISSN: 1098-3627 (online)

DOI: http://dx.doi.org/10.1090/conm/610

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10 9 8 7 6 5 4 3 2 1   19 18 17 16 15 14

Perspectives in Representation Theory

This volume is dedicated to Igor Frenkel on the occasion of his sixtieth birthday.

Preface

This is the proceedings of the conference *Perspectives in representation theory*, held at Yale University May 12–17, 2012, in honor of the 60th birthday of Prof. Igor Frenkel. It contains papers by some of the speakers at the conference and their collaborators, as well as a summary of the work of Igor Frenkel, prepared by his former students and collaborators. The meeting featured talks by mathematicians working in representation theory, with an emphasis on its relations to other subjects (notably, topology, algebraic geometry, number theory, and mathematical physics). The papers in this volume concern current progress on the following (interrelated) topics: vertex operator algebras and chiral algebras, conformal field theory, the (geometric) Langlands program, affine Lie algebras, Kac-Moody algebras, quantum groups, crystal bases and canonical bases, quantum cohomology and K-theory, geometric representation theory, categorification, higher-dimensional Kac-Moody theory, integrable systems, quiver varieties, representations of real and p-adic groups, quantum gauge theories. The conference was an occasion to discuss representation theory in the context of its connections with numerous other subjects, and to discuss some of the most recent advances in representation theory, including those which occurred thanks to application of techniques in other areas of mathematics, and of ideas of quantum field theory and string theory.

Videos of most of the talks given at the conference, as well as photos taken during the event, can be found on the conference website at:

`www.math.yale.edu/frenkel60`

The organizing committee consisted of

P. Etingof (M.I.T.)
M. Khovanov (Columbia University)
A. Kirillov Jr. (SUNY Stony Brook)
A. Lachowska (Yale University)
A. Licata (Australian National University)
A. Savage (University of Ottawa)
G. Zuckerman (Yale University)

The organizers of the conference would like to thank the National Science Foundation and Yale University for their support, without which the event would not have been possible.

Pavel Etingof
Mikhail Khovanov
Alistair Savage

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Contemporary Mathematics
Volume **610**, 2014
http://dx.doi.org/10.1090/conm/610/12155

On the work of Igor Frenkel

John Duncan, Pavel Etingof, Ivan Ip, Mikhail Khovanov, Matvei Libine, Anthony Licata, Alistair Savage, and Michael Schlosser

Introduction
by Pavel Etingof

Igor Frenkel is one of the leading representation theorists and mathematical physicists of our time. Inspired by the mathematical philosophy of Herman Weyl, who recognized the central role of representation theory in mathematics and its relevance to quantum physics, Frenkel made a number of foundational contributions at the juncture of these fields. A quintessential mathematical visionary and romantic, he has rarely followed the present day fashion. Instead, he has striven to get ahead of time and get a glimpse into the mathematics of the future – at least a decade, no less. In this, he has followed the example of I. M. Gelfand, whose approach to mathematics has always inspired him. He would often write several foundational papers in a subject, and then leave it for the future generations to be developed further. His ideas have sometimes been so bold and ambitious and so much ahead of their time that they would not be fully appreciated even by his students at the time of their formulation, and would produce a storm of activity only a few years later. And, of course, as a result, many of his ideas are still waiting for their time to go off.

This text is a modest attempt by Igor's students and colleagues of various generations to review his work, and to highlight how it has influenced in each case the development of the corresponding field in subsequent years.

1. Representation theory of affine Lie algebras
by Alistair Savage and Anthony Licata

Among infinite-dimensional Lie algebras, it is the theory of affine Lie algebras that is the richest and most well understood. Igor Frenkel's contributions to this subject are both numerous and diverse, and his are among the deepest and most fundamental developments in the subject. These contributions began in his 1980 Yale University thesis, the core of which was later published in the paper [**Fth**]. In his thesis, Frenkel adapts the orbital theory of A. A. Kirillov to the setting of affine Lie algebras, giving, in particular, a formula for the characters of irreducible highest weight representations in terms of orbital integrals. The technical tools required for Frenkel's orbital theory include a tremendous amount of interesting mathematics, including "the Floquet theory of linear differential equations with periodic coefficients, the theory of the heat equation on Lie groups, the theories of Gaussian and Wiener measures, and of Brownian motion." (Quote from the MathSciNet review of [**Fth**].) Thus Frenkel's thesis gives one of the early examples of a central theme in the theory of affine Lie algebras, namely, the rich interaction between their representation theory and the rest of mathematics.

A fundamental contribution of Frenkel to infinite-dimensional representation theory came in his joint paper with Kac [**FK80**]. In this paper, the authors formally introduced vertex operators into mathematics, and used them to give an explicit construction of the basic level one irreducible representation of a simply-laced affine Lie algebra. (A very similar construction was given independently around the same time by Segal [**S**].) In important earlier work, Lepowsky-Wilson [**LW**] gave a twisted construction of the basic representation for $\widehat{\mathfrak{sl}_2}$, and this twisted construction was then generalized to other types by Kac-Kazhdan-Lepowsky-Wilson [**KKLW**]. Vertex operators themselves had also been used earlier in the dual resonance models of elementary particle physics. But it was the ground-breaking paper of Frenkel and Kac that developed their rigorous mathematical foundation, and established a direct link between vertex operators and affine Lie algebras. Thus began the mathematical subject of vertex operator algebras, a subject which has had profound influence on areas ranging from mathematical physics to the study of finite simple groups. Frenkel also gave closely related spinor constructions of fundamental representations of affine Lie algebras of other types in [**FPro**].

Another important example of Frenkel's work at the interface of affine Lie algebras and mathematical physics is his work on the boson-fermion correspondence [**Fre81**]. In the course of establishing an isomorphism between two different realizations of simply-laced affine Lie algebras, he realized that his result could be reformulated in the language of quantum field theory, implying an equivalence of physical models known to physicists as the *boson-fermion correspondence*. This paper was the first on the connection between infinite dimensional Lie algebras and 2d conformal field theory. Also, in [**FF85**], Feingold and Frenkel obtained bosonic and fermionic constructions of all classical affine Lie algebras. Further related but independently important developments appeared in [**Fre85**] and in [**Flr**], where Frenkel established what is now known as *level-rank duality* for representations of affine Lie algebras of type A, and obtained upper bounds for root multiplicities for hyperbolic Kac-Moody algebras applying the no-ghost theorem from physics. In another paper with Feingold, [**FF83**], Frenkel suggested a relation between hyperbolic Kac-Moody algebras and Siegal modular forms, which was further studied in the works of Borcherds and Gritsenko-Nikulin.

The relevance of affine Lie algebras and their representation theory was highlighted by Frenkel in his invited address, entitled "Beyond affine Lie algebras", at the 1986 ICM in Berkeley ([**FBa**]). Since then, his foundational work in and around the subject of affine Lie algebras has been extremely influential in other areas, perhaps most notably in vertex algebra theory, in the representation theory of quantum groups, and in geometric representation theory and categorification. Frenkel's work on affine Lie algebras comprises his first major contributions to mathematics, and the fundamental nature of this work has been repeatedly confirmed by the relevance of affine Lie algebras and their representation theory in both mathematics and mathematical physics.

2. Quantum Knizhnik-Zamolodchikov equations by Pavel Etingof

In 1984 Knizhnik and Zamolodchikov studied the correlation functions of the Wess-Zumino-Witten (WZW) conformal field theory, and showed that they satisfy a remarkable holonomic system of differential equations, now called the Knizhnik-Zamolodchikov (KZ) equations. Soon afterwards Drinfeld and Kohno proved that the monodromy representation of the braid group arising from the KZ equations is given by the R-matrices of the corresponding quantum group, and Schechtman and Varchenko found integral formulas for solutions of the KZ equations. At about the same time, Tsuchiya and Kanie proposed a mathematically rigorous approach to the WZW correlation functions, by using intertwining operators between a Verma module over an affine Lie algebra and a (completed) tensor product of a Verma module with an evaluation module:

$$\Phi(z) : M_{\lambda,k} \longrightarrow M_{\mu,k}\hat{\otimes}V(z).$$

Namely, they proved that highest matrix elements of products of such operators (which are the holomorphic parts of the correlation functions of the WZW model) satisfy the KZ equations. This construction can be used to derive the Drinfeld-Kohno theorem, as it interprets the monodromy of the KZ equations in terms of the exchange matrices for intertwining operators $\Phi(z)$, which are twist equivalent (in an appropriate sense) to the R-matrices of the quantum group.

This set the stage for the pioneering paper by I. Frenkel and N. Reshetikhin [**FR**], which was written in 1991 (see also the book [**EFK**] based on lectures by I. Frenkel, which contains a detailed exposition of this work). In this ground-breaking work, Frenkel and Reshetikhin proposed a q-deformation of the theory of WZW correlation functions, KZ equations, and their monodromy, and, in effect, started the subject of q-deformed conformal field theory, which remains hot up to this day [1]. Namely, they considered the intertwining operators $\Phi(z)$ for quantum affine algebras, and showed that highest matrix elements of their products, $\langle\Phi_1(z_1)....\Phi_n(z_n)\rangle$, satisfy a system of difference equations, which deform the KZ equations; these equations are now called the quantum KZ equations. They also showed that the monodromy of the quantum KZ equations is given by the exchange matrices for the quantum intertwining operators, which are elliptic functions of z, and suggested that such matrices should give rise to "elliptic quantum groups".

This work had a strong influence on the development of represenatation theory in the last 20 years, in several directions.

First of all, the quantum KZ equations arose in several physical contexts (e.g., form factors of F. Smirnov, or solvable lattice models considered by Jimbo, Miwa, and their collaborators).

Also, Felder, Tarasov, and Varchenko, building on the work of Matsuo, generalized the Schectman-Varchenko work to the q-case, and found integral formulas for solutions of the quantum KZ equations.

At about the same time, G. Felder proposed the notion of elliptic quantum groups based on the dynamical Yang-Baxter equation, which is satisfied by the exchange matrices. This theory was further developed by Felder, Tarasov, and Varchenko, and also by Etingof-Varchenko, who proposed a theory of dynamical

[1] We note that q-deformation of some structures of conformal field theory, namely the vertex operator construction of [**FK80**], was already considered in an earlier paper, [**FJ88**].

quantum groups and dynamical Weyl groups (generalizing to the q-case the theory of Casimir connections).

Another generalization of the quantum KZ equation, corresponding to Weyl groups, was considered by Cherednik, and this generalization led to his proof of Macdonald's conjectures and to the discovery of double affine Hecke algebras, also called Cherednik algebras, which are in the center of attention of representation theorists in the past 15 years.

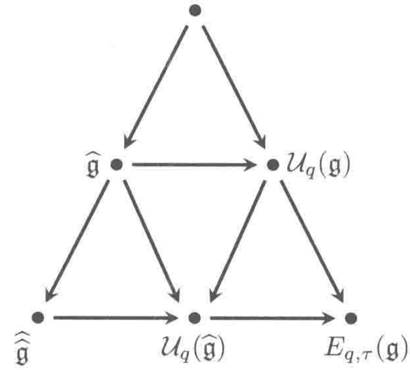
Yet another generalization is the theory of elliptic quantum KZ equations (or quantum Knizhnik-Zamolodchikov-Bernard equations), which was developed in the works of Etingof, Felder, Schiffmann, Tarasov, and Varchenko.

The paper [**FR**] also served as a motivation for Etingof, Schedler, and Schiffmann in their consruction of explicit quantization of all non-triangular Lie bialgebra structures on simple Lie algebras (classified by Belavin and Drinfeld) and to Etingof and Kazhdan in their work on quantization of Lie bialgebras associated to curves with punctures.

Finally, the ideas of this paper played an important role in the work of Etingof and Kirillov Jr. on the connection between Macdonald polynomials and quantum groups, and their definition of affine Macdonald polynomials, and in a generalization of this work by Etingof and Varchenko (the theory of traces of intertwining operators for quantum groups). These structures and functions are now arising in algebraic geometry (e.g. the work of A. Negut on integrals over affine Laumon spaces). Also, quantum KZ equations and q-Casimir connections are expected to arise in the study of quantum K-theory of quiver varieties.

3. Double loop groups by Pavel Etingof

Around 1990, when the loop algebra/quantum group revolution of the 1980s and early 1990s had reached its culmination, Igor Frenkel suggested that the next important problem was to develop a theory of double loop algebras. More specifically, he proposed a philosophy of three levels in Lie theory (and thereby in mathematics in general), illustrated by the following diagram:



In this diagram, the left downward arrows stand for affinization (taking loops), and the right downward arrows stand for quantization (q-deformation). The first level represents "classical" Lie theory, i.e., the structure and representation theory of complex semisimple Lie groups and Lie algebras. The second level represents affine Lie algebras and quantum groups, i.e., structures arising in 2-dimensional conformal and 3-dimensional topological field theory. The connection between them, depicted by the horizontal arrow, is the Drinfeld-Kohno theorem on the monodromy of the KZ equations, which is a part of the Kazhdan-Lusztig equivalence of categories. Finally, the third level is supposed to represent double affine Lie algebras, quantum affine algebras, and double (or elliptic) quantum groups. These three levels are supposed to correspond to discrete subgroups of the complex plane of ranks 0,1,2, respectively, and higher levels are not supposed to exist in the same sense because there are no discrete subgroups of $\mathbf{C}$ of rank > 2. [2]

At the time this philosophy was formulated, there wasn't much known about the third level of the diagram. Specifically, while quantum affine algebras were being actively studied, and Igor Frenkel's work with Reshetikhin on quantum KZ equations (subsequently developed by Felder, Tarasov, Varchenko, and others) shed a lot of light on what elliptic quantum groups and the quantum Drinfeld-Kohno theorem should be, the left lower corner of the diagram – the double loop algebras – remained mysterious. Yet, Igor insisted that this corner is the most important one, and that the study of double loops holds a key to the future of representation theory.

To develop the theory of double loop groups following the parallel with ordinary loop groups, one has to start with central extensions. This direction was taken up in our joint paper [**EtF**], where we constructed the central extension of the group of maps from a Riemann surface to a complex simple Lie group by the Jacobian of this surface (i.e., for genus 1, by an elliptic curve), and showed that the coadjoint orbits of this group correspond to principal G-bundles on the surface. This work was continued in the paper [**FrKh**], which extends to the double loop case the Mickelsson construction of the loop group extension by realizing the circle as a boundary of a disk, and then realizing a union of two such disks as a boundary of a ball. Namely, the circle is replaced by a complex curve (Riemann surface), the disk by a complex surface, and the ball by a complex threefold; then a similar formula exists, in the context of Leray's residue theory instead of De Rham theory. This work led to subsequent work by Khesin and Rosly on polar homology, as well as to the work of Frenkel and Todorov on a complex version of Chern-Simons theory, [**FTo**]. In this latter work, they start to develop the complex version of knot theory, in which the role of the 3-sphere is played by a Calabi-Yau threefold, and the role of the circle is played by a complex curve. In particular, these works led to a definition of the holomorphic linking number between two complex curves in a Calabi-Yau threefold, which is a complex analog of the classical Gauss linking number, previously studied by Atiyah in the case of $\mathbb{CP}^1$.

In spite of this progress, however, it is still not clear what the representation theory of central extensions of double loop groups should be like. Perhaps we don't yet have enough imagination to understand what kind of representations (or maybe analogous but more sophisticated objects) we should consider, and this is a problem for future generations of mathematicians.

[2] I must admit that initially I did not take this philosophy too seriously, and we used the diagram in a Holiday party skit. However, with time it acquired quite a few concrete mathematical incarnations, and, ironically, defined much of my own work.

4. Vertex operator algebras
by John Duncan

The *(normalised) elliptic modular invariant*, denoted $J(\tau)$, is the unique $SL_2(\mathbb{Z})$-invariant holomorphic function on the upper-half plane $\mathbb{H}$ with the property that $J(\tau) = q^{-1} + O(q)$ for $q = e^{2\pi i\tau}$. In the late 1970's McKay and Thompson made stunning observations relating the coefficients of the Fourier expansion

$$J(\tau) = q^{-1} + \sum_{n>0} c(n)q^n \tag{1}$$

of $J(\tau)$ to the dimensions of the irreducible representations of the (then conjectural) Monster sporadic simple group. This led to the conjecture [**Tho79b**] that there is a naturally defined infinite-dimensional representation

$$V = V_{-1} \oplus V_1 \oplus V_2 \oplus \cdots \tag{2}$$

for the Monster group with the property that $\dim V_n = c(n)$. Consideration [**Tho79a**] of the functions $T_g(\tau)$ obtained by replacing $\dim V_n = \mathrm{tr}|_{V_n} e$ with $\mathrm{tr}|_{V_n} g$ for g in the Monster led to the birth of *monstrous moonshine* and the *monstrous game* of Conway–Norton [**CN79**]. Thus, for the elucidation of monstrous moonshine, it became an important problem to construct such a representation—a *moonshine module* for the monster—explicitly. Igor Frenkel's pioneering work on vertex representations of affine Lie algebras, such as appears in [**FK80, Fre81, Fre85, FF85**], furnished important foundations for the work [**FLM84, FLM85, FLM88**] that would eventually realise this goal.

In [**Gri82**] Griess constructed the Monster group explicitly as the automorphism group of a certain commutative non-associative finite-dimensional algebra, thereby establishing its existence. The great insight of Frenkel–Lepowsky–Meurman was to recognise this algebra as a natural analogue of a simple finite dimensional complex Lie algebra $\mathfrak{g}$, viewed as a subalgebra of its affinization $\hat{\mathfrak{g}}$. Identifying Griess's algebra with (a quotient of) V_1 they attached vertex operators to the elements of this space and used them in [**FLM84, FLM85**] (see also [**FLM88**]) to recover the Griess algebra structure. In this way the non-associativity of the finite-dimensional Griess algebra was replaced with the associativity property of vertex operators.

The Frenkel–Lepowsky–Meurman construction [**FLM84, FLM85**] of the moonshine module V utilised the Leech lattice in much the same way as the root lattice of a Lie algebra of ADE type had been used to construct its basic representation in [**FK80**], but an important twisting procedure was needed in order to ensure the vanishing of the subspace V_0 in (2). This procedure (realised in full detail in [**FLM88**]) turned out to be the first rigorously constructed example of an *orbifold conformal field theory* and thus represented a significant development for mathematical physics.

Building upon the work of [**FLM84, FLM85**], Borcherds discovered a natural way to attach vertex operators to all elements of V, and several other examples, in [**Bor86**] and used this to define the notion of *vertex algebra*, which has subsequently met many important applications in mathematics and mathematical physics. The closely related notion of *vertex operator algebra (VOA)* was introduced in [**FLM88**]. A VOA comes equipped with a representation of the Virasoro algebra, and this hints at the importance of VOAs in conformal field theory. The central charge of the