LIE GROUPS

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PREFACE

The theory of Lie groups rests on three pillars: analysis, topology and algebra. Correspondingly it is possible to distinguish several phases, overlapping in some degree, in its development. It also allows one to regard the subject from different points of view, and it is the algebraic standpoint which has been chosen in this tract as the most suitable one for a first introduction to the subject.

The aim has been to develop the beginnings of the theory of Lie groups, especially the fundamental theorems of Lie relating the group to its infinitesimal generators (the Lie algebra); this account occupies the first five chapters. Next to Lie's theorems in importance come the basic properties of subgroups and homomorphisms, and they form the content of Chapter VI. The final chapter, on the universal covering group, could perhaps be most easily dispensed with, but, it is hoped, justifies its existence by bringing back into circulation Schreier's elegant method of constructing covering groups.

Of course whatever outlook is adopted, it is necessary to have a number of tools at one's disposal, and these have been provided in the book as far as possible. Thus before we come to Lie groups proper, the notions of analytic manifold and topological group are introduced. Lie algebras and exterior algebras are brought in later as they are needed, while theorems from analysis, such as the existence theorem for the solutions of total differential equations and the implicit function theorem, are proved in an appendix. It has been assumed that the reader has some knowledge of algebra and topology, but this need only include the elementary properties of groups and vector spaces, and the elementary notions of analytic topology.

This book owes a great deal to my colleagues in Manchester; when I gave a course on the subject in 1954, their comments showed me how much I had still to learn, and I had some opportunity of doing so in subsequent discussions with them. In particular, Dr Graham Higman and Mr G. E. H. Reuter, with

their advice and comments on the earlier parts of the manuscript, saved me from a number of errors. Dr J. A. Green read the whole manuscript and made many valuable suggestions, and Dr P. J. Hilton read parts of the manuscript including the last chapter, which was much improved as a result. To all of them I should like to express my gratitude.

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CONTENTS

INTRODUCTION

The theory of *continuous groups*, or as they are now called, *Lie groups*, was developed by Sophus Lie (1842–99) in connexion with the integration of systems of differential equations. These groups first arose as groups of transformations, while now they are considered just as groups in the abstract. This is similar to the situation in algebra where groups themselves appeared as permutation groups before they came to be regarded as abstract groups.

We can see how such a transformation group arises by considering the system of differential equations

$$\frac{dx_i}{dt} = u_i(x) \quad (i = 1, \ldots, n), \tag{1}$$

where x_i are the Cartesian coordinates of a point x in real Euclidean n-space. Taking $n = 3$, we can interpret the system (1) as follows: We have a fluid moving in space and the velocity of the particle of fluid at the point $x = (x_i)$ has the components $u_i(x)$. Let us consider the particle of fluid which at time $t = 0$ is at x and ask for its position at some subsequent time $t > 0$. The answer is obtained by integrating the equations (1), and is of the form

$$x'_i = f_i(x, t). \tag{2}$$

We can think of (2) as defining a transformation of the whole of space; with each point x we associate the point x' which is reached by the fluid initially at x after a time t. We express this by writing

$$x' = xS_t;$$

thus (S_t) is a family of transformations of space, and it is easy to see that these transformations form a group (if we assume that the equations (1) can be integrated for all x):

$$xS_tS_{t'} = xS_{t+t'}, \quad xS_0 = x, \quad xS_tS_{-t} = x \quad \text{(for any point } x\text{)}.$$

This group of transformations, regarded as an abstract group, is isomorphic to the additive group of real numbers.†

† Provided that the fluid never returns to its initial state (see 2.9, in particular Theorem 2.9.3).

But the way in which it arose emphasizes some special features of it:

(i) The suffix of the transformation which corresponds to the product $S_t S_{t'}$, is given by the function

$$\phi(t, t') = t + t'$$

of two variables which is continuous and even differentiable.

(ii) If we write (1) in the form

$$\delta x_i = u_i(x)\, \delta t,$$

we can consider it as an 'infinitesimal transformation'

$$x_i \rightarrow x_i + \delta x_i, \tag{3}$$

and we may think of each transformation S_t as built up by iterating the infinitesimal transformation (3). This shows that in some sense our group may be thought of as the analogue of a cyclic group, with (3) as its (infinitesimal) generator.

(iii) In order to obtain a group of transformations we had to postulate that the equations (1) have a solution for all values of x. In general, this may not be so, and we then obtain, not a group of transformations, but only a portion of a group.

This indicates the features which we shall expect of a Lie group. Thus it must be possible to introduce a coordinate system into the group, or at least part of it, such that the multiplication law of the group is expressed in terms of this coordinate system by differentiable functions. The most convenient way of taking account of the fact that this coordinate system may not be everywhere defined is to postulate that our group is a topological space in which the group operations are continuous, i.e. a *topological group*, and that a coordinate system is defined on some neighbourhood of the unit element. One can then show that although there need not be a coordinate system for the whole group, at least there is one defined on a neighbourhood of each point.

In a topological group, as in every topological space, one can distinguish between local properties, such as being locally compact or locally connected, and global properties which refer to the space as a whole. Moreover, in a Lie group one can go beyond the local to the infinitesimal. Thus one can show that a Lie group of transformations is generated by a finite number of infinitesimal transformations. In case the Lie group is given abstractly, we simply regard it as a transformation group by letting it act on itself by right translations. If the multiplication law is given by n differentiable functions $\phi_i(x, y)$, representing the coordinates of the product xy, then the general infinitesimal right translation is obtained by expanding the ϕ's in a Taylor series with respect to the y's and neglecting powers higher than the first.

To consider infinitesimal transformations instead of the finite transformations is to linearize the problem. Thus to the product of two elements of the group corresponds the sum of two infinitesimal transformations; so the latter form a linear space. Another operation is obtained by considering the commutator $x^{-1}y^{-1}xy$ of two elements of the group. This corresponds to a kind of product of infinitesimal transformations, and relative to this product the infinitesimal transformations form a non-associative linear algebra which is known as the *Lie algebra* of the group. This algebra is perhaps most familiar in the case of the group of rotations in 3-dimensional Euclidean space. It is well known that the infinitesimal rotations in space can be represented by vectors, and with this convention the usual vector product is just the multiplication for which the vectors form the Lie algebra of the rotation group.

One of the basic achievements of Lie's theory was to determine a set of conditions satisfied by the Lie algebra of a Lie group and to show that any linear algebra satisfying these conditions belongs to a Lie group.† It is this part of the theory, establishing the connexion between Lie groups and Lie algebras, together with the more fundamental properties of Lie groups, which we shall present here.

† Lie himself only considered this problem locally, i.e. he was only concerned with constructing a portion of the group near the unit element. The final step of embedding any such 'local' Lie group in a 'global' group is based on a comparatively recent theorem and belongs more to the theory of Lie algebras. For this reason a proof of this result has not been included (see Chapters VI, VII).

CHAPTER I

ANALYTIC MANIFOLDS

1.1. Charts and coordinates. One of the basic concepts to be used is that of the n-dimensional real Euclidean space. We denote this space by R^n, and indicate the coordinates of a point by attaching superscripts. Thus if $x \in R^n$, the coordinates of x are written $x^1, x^2, \ldots, x^n$, or more briefly x^i $(i = 1, \ldots, n)$. By means of the usual metric on R^n

$$d(x, y) = \{\sum_i (x^i - y^i)^2\}^{\frac{1}{2}},$$

a topology is defined on R^n, which allows us to regard it as a Hausdorff space.†

More generally, we shall consider spaces which behave locally like R^n. Thus consider a topological space T and let W be a non-empty open subspace of T which is homeomorphic to an open subspace X of R^n. If σ: $p \to p^\sigma$ denotes a homeomorphism of W onto X, we call σ a *chart in* T, or, more precisely, *on* W. In a given chart σ on W, each point p of W corresponds to a point $x = p^\sigma$ of R^n so that p may be described by x^i, the coordinates of x. The numbers x^i are called the *coordinates of* p (in the chart σ) and n is the *dimension* of the chart.

Now suppose that there is a homeomorphism Φ of X onto another open subspace Y of R^n and let Ψ be its inverse. If $y = x^\Phi$ is the general point of Y, with coordinates y^i $(i = 1, \ldots, n)$, then Φ and Ψ may be described by means of continuous functions ϕ^i and ψ^i:

$$\left.\begin{aligned} y^i &= \phi^i(x^1, \ldots, x^n) \\ x^i &= \psi^i(y^1, \ldots, y^n) \end{aligned}\right\} \quad (i = 1, \ldots, n).$$

Occasionally we shall use the letter x to denote the set of coordinates $(x^1, \ldots, x^n)$ as well as the point of R^n which they represent. Then the above equations of transformation between x and y may be written

$$\left.\begin{aligned} y^i &= \phi^i(x) \quad (x \in X), \\ x^i &= \psi^i(y) \quad (y \in Y). \end{aligned}\right\} \qquad (1)$$

† For the topological concepts used, see, for example, Bourbaki[1].

If we combine the mappings σ and Φ from W to Y, we obtain the homeomorphism $\sigma\Phi$: $p \to (p^\sigma)^\Phi$ of W onto Y, so that $\sigma\Phi$ is again a chart on W.

Conversely, if σ and τ are any two charts on the same subspace W of T, mapping W into X and Y respectively, then $\Phi = \sigma^{-1}\tau$ is a homeomorphism of X onto Y with inverse $\Psi = \tau^{-1}\sigma$, so that the coordinates x and y of corresponding points in X and Y are related by equations of the form (1). We may regard the passage from x to y as a change of coordinates, and what has been said shows that the equations (1) (with continuous functions ϕ^i and ψ^i) are the most general equations describing a change of coordinates.†

Two charts in T whose coordinates are related by the equations (1) are said to be *analytically related* at a point p of T, if they are defined on a neighbourhood‡ of p, and if the functions ϕ^i, ψ^i occurring in (1) are analytic functions of their arguments at p^σ and p^τ respectively. Here a function $f(x)$ is said to be *analytic* at the point a of R^n if it can be expressed as a convergent power series in $x^i - a^i$ $(i = 1, \ldots, n)$ in some neighbourhood of the point a. If two charts are analytically related at every point of T at which both are defined, we say that they are *analytically related*. This is true in particular if there is no point at which both charts are defined.

1.2. Analytic structures.

A topological space T is said to be *locally Euclidean* at a point p, if there exists a chart σ on a neighbourhood of p; we then say that σ is a chart *at* p. A Hausdorff space which is locally Euclidean at each point is called a *manifold*. Thus in a manifold M each point has a chart defined on some neighbourhood, a property which may be expressed by saying that the family of all charts in M *covers* M.

† It follows from the theorem on the invariance of the dimension that two charts on the same set have the same dimension (see, for example, Hurewicz, W. and Wallman, H., *Dimension Theory*, Princeton, 1941). For the particular case with which we are concerned—that of analytically related charts—this will be proved directly in 1.4.

‡ We use the term 'neighbourhood' in the sense of Bourbaki: A neighbourhood of a point p in a topological space T is a subset of T which contains p in its interior.

In this definition *all* the charts in M were admitted. We now restrict the class of charts in order to obtain a more specific structure.

DEFINITION. Let M be a Hausdorff space. Then an *analytic structure* on M is a family $\mathscr{F}$ of charts defined in M such that

M. 1. *At each point of M there is a chart which belongs to $\mathscr{F}$.*

M. 2. *Any two charts of $\mathscr{F}$ are analytically related.*

M. 3. *Any chart in M which is analytically related to every chart of $\mathscr{F}$ itself belongs to $\mathscr{F}$.*

We shall express M. 2 and M. 3 by saying that $\mathscr{F}$ is *analytic* and *maximal*, respectively. Thus an analytic structure on M is a maximal analytic family of charts covering M. It is clear that a Hausdorff space with an analytic structure is necessarily a manifold, and the space, together with this structure, is called an *analytic manifold*. By a chart in an analytic manifold we always understand a chart belonging to the analytic structure. When we wish to stress this fact we refer to the members of the structure as *admissible* charts.

In practice it is usually impossible to obtain a maximal analytic family covering a space by an explicit construction. This difficulty is overcome by the following theorem which shows that it is sufficient to construct any analytic family covering the space.

THEOREM 1.2.1. *Let M be a Hausdorff space and $\mathscr{C}$ an analytic family of charts which covers M. Then there is a uniquely determined maximal analytic family of charts which contains $\mathscr{C}$.*

Proof. Let $\mathscr{F}$ be the set of all charts in M which are analytically related to each member of $\mathscr{C}$; we shall show that $\mathscr{F}$ has the required properties. Let us express the fact that two charts σ and τ are analytically related at p by writing $\sigma \underset{p}{\sim} \tau$. If ρ, σ and τ are any charts at p, then it is easily verified that $\rho \underset{p}{\sim} \rho$, that $\rho \underset{p}{\sim} \sigma$ implies $\sigma \underset{p}{\sim} \rho$, and that $\rho \underset{p}{\sim} \sigma$, $\sigma \underset{p}{\sim} \tau$ imply $\rho \underset{p}{\sim} \tau$. Thus '$\underset{p}{\sim}$' is an equivalence relation on the set of charts at p. Now let ρ_1 and ρ_2 be any members of $\mathscr{F}$ and let W be the intersection of the sets on which ρ_1 and ρ_2 are defined. As an intersection of open sets W is again open. If W is empty, then ρ_1 and ρ_2 are analytically related by definition. Otherwise let p be any point of W; since W

is open it is a neighbourhood of p, and since $\mathscr{C}$ covers M, there is a chart σ at p which belongs to $\mathscr{C}$. By definition of $\mathscr{F}$, $\rho_1 \underset{p}{\sim} \sigma$ and $\rho_2 \underset{p}{\sim} \sigma$, whence $\rho_1 \underset{p}{\sim} \rho_2$. Thus ρ_1 and ρ_2 are analytically related at each point p of W, and hence they are analytically related. This proves that $\mathscr{F}$ is analytic. Clearly $\mathscr{F} \supseteq \mathscr{C}$, and if τ is analytically related to each member of $\mathscr{F}$ then it is analytically related to each member of $\mathscr{C}$ and hence belongs to $\mathscr{F}$. Thus $\mathscr{F}$ is a maximal analytic family containing $\mathscr{C}$. If $\mathscr{F}_1$ is another maximal analytic family containing $\mathscr{C}$, then each member of $\mathscr{F}_1$ is analytically related to each member of $\mathscr{C}$ and therefore belongs to $\mathscr{F}$. Hence $\mathscr{F}_1 \subseteq \mathscr{F}$, and similarly $\mathscr{F} \subseteq \mathscr{F}_1$, which proves that $\mathscr{F}_1 = \mathscr{F}$. Thus $\mathscr{F}$ is unique and the proof is complete.

The family $\mathscr{F}$ in Theorem 1.2.1 covers M (since $\mathscr{C}$ does) and therefore defines an analytic structure on M. So in order to define an analytic structure on a space M it is enough to specify an analytic family of charts which covers M. Of course there may be different analytic families covering M which define the same analytic structure. The necessary and sufficient condition for this to be the case is given by the

COROLLARY. *Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two analytic families of charts covering a space M. Then there is a maximal analytic family containing $\mathscr{C}_1$ and $\mathscr{C}_2$ if and only if for each point p of M there is a chart of $\mathscr{C}_1$ which is analytically related at p to a chart of $\mathscr{C}_2$.*

The condition is clearly necessary. Conversely, if it is satisfied, then by the argument used to prove Theorem 1.2.1, every chart of $\mathscr{C}_1$ is analytically related to every chart of $\mathscr{C}_2$ and hence the family $\mathscr{C}$ of all charts belonging to $\mathscr{C}_1$ or $\mathscr{C}_2$ is analytic. If $\mathscr{F}$ is the maximal analytic family containing $\mathscr{C}$, then $\mathscr{F} \supseteq \mathscr{C}_1$ and $\mathscr{F} \supseteq \mathscr{C}_2$; this proves the corollary.

As an example let us consider the surface of a unit sphere S in three dimensions. At any point p on S take a great circle C through p and take a system of latitude and longitude with C as the equator and p as defining the 'Greenwich meridian'. If the poles of S with respect to C ('north and south poles') are joined by a line l not passing through p (the 'date-line'), then the complement of l is a neighbourhood of p on which latitude and longitude define a chart. If the same construction is carried out for another point q of S, then the two charts are analytically related. This is easily verified by choosing the centre of S to be at the origin (0, 0, 0), the point p at (1, 0, 0) and by observing that the Cartesian coordinates of the general point of S in terms of latitude θ and longitude ϕ at p are $(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$. By equating these expressions to the corresponding expressions in terms of the chart at q we obtain analytic relations which can be solved for either set. Thus we have an analytic family of charts covering S, and by Theorem 1.2.1 this defines an analytic structure on S.

Another analytic family covering S may be obtained as follows: We take $p \in S$ to be the north pole and consider the stereographic projection from the south pole on the plane through the equator. This is a homeomorphism of the punctured sphere (namely, the sphere with the south pole removed) and the Euclidean plane, and hence defines a chart at p. In order to obtain the coordinates of the general point q in this chart, we map it into q_0, the point in which the straight line from the south pole $(0, 0, -1)$ to q cuts the (x, y)-plane. If the Cartesian coordinates are (x, y, z), its coordinates in the chart are $\left(\frac{x}{1+z}, \frac{y}{1+z}\right)$, the plane coordinates of q_0. It is again not hard to verify that if such a chart is constructed at each point of S, then these charts form a second analytic family. Moreover, the charts in these two families are analytically related. Hence, by the corollary to Theorem 1.2.1, these two families are contained in the same maximal analytic family and therefore define the same analytic structure.

We note the following examples of analytic manifolds:

1. The space R^n. The Cartesian coordinates in R^n serve as a chart at every point. We shall denote the analytic manifold so defined by $\mathfrak{R}^n$; for $\mathfrak{R}^1$ we shall also write $\mathfrak{R}$.

2. The torus T^n. This is the subspace of the n-dimensional complex Euclidean space described by

$$z_\nu = \exp 2\pi i\theta_\nu \quad (0 \leqslant \theta_\nu < 1). \tag{2}$$

Topologically the torus is the Cartesian product of n circles. In particular, for $n=1$ we obtain a circle, and for $n=2$ the familiar anchor ring. The formula (2) can be used to define an analytic structure on T^n, and the analytic manifold so obtained is denoted by $\mathfrak{T}^n$. We also write $\mathfrak{T}$ instead of $\mathfrak{T}^1$ and sometimes refer to $\mathfrak{T}$ as the (analytic manifold of) real numbers mod 1.

3. The set $GL(n, R)$ of all automorphisms of a vector space V of dimension n over R, that is, the general linear group. In terms of a given basis of V the automorphisms may be expressed as non-singular $n \times n$ matrices with coefficients in R, and the n^2 coefficients serve as a chart at each point of $GL(n, R)$.

4. A single point, or, more generally, any discrete space,† may be regarded as a 'zero-dimensional' analytic manifold.

Ex. Show that there are distinct analytic structures on R which induce the same topology (consider the coordinate transformation $y = x^3$).

1.3. Real functions on a manifold.

Let M be a manifold and f a real-valued function defined on a part (possibly the whole) of M. We shall express this by saying that f is defined *in* M. If σ is a chart on some subset W of M on which f is defined, then we can express f as a function of n real variables by writing

$$\bar{f}(x) = f(p), \tag{3}$$

where $x = p^\sigma$. If τ is another chart on W which is related to σ by (1), then we can express f similarly in terms of τ: $f(p) = \bar{\bar{f}}(y)$ $(y = p^\tau)$. It is clear that $\bar{f}$ and $\bar{\bar{f}}$ are related by the equations

$$\left.\begin{aligned} \bar{f}(x) &= \bar{\bar{f}}(\phi(x)), \\ \bar{\bar{f}}(y) &= \bar{f}(\psi(y)), \end{aligned}\right\} \tag{4}$$

which hold identically in x and y.

In the foregoing discussion, where only two charts occurred, it was more convenient to denote the coordinates of the general point p in these charts by x and y respectively instead of p^σ and p^τ. We shall adopt this practice generally and even refer to a given chart by naming the coordinate functions which it defines, rather than by its proper name. For distinction we enclose the symbol for the coordinates in brackets; thus in future we shall usually speak of charts (x), (y), ... and not σ, τ, ...

† A topological space is said to be *discrete*, if any subset consisting of a single point is open.

A real-valued function f defined in M is a mapping of one topological space into another, and so we know what it means for f to be continuous. From the definitions in 1.1 and 1.2 we see that f is continuous at a point p if and only if its expression in terms of a chart at p is a continuous function of its n arguments at p. If $\mathfrak{M}$ is an analytic manifold, we shall say that a real-valued function f in $\mathfrak{M}$ is *analytic* at a point p if it is defined on some neighbourhood of p and its expression in terms of an admissible chart σ at p is an analytic function of its arguments at p^σ. It is easily verified that this definition does not depend on the choice of the chart σ. A function which is analytic at every point at which it is defined is called *analytic*. We shall denote by $\mathscr{A}_p$ the set of analytic functions which are defined at p, and by $\mathscr{A}$ the set of all analytic functions in $\mathfrak{M}$.

Ex. 1. An analytic function of an analytic function is analytic.

Ex. 2. Each coordinate of an admissible chart is analytic.

1.4. Tangent vectors. Let $\mathfrak{M}$ be an analytic manifold, p a point of $\mathfrak{M}$ and (x) a chart at p (understood to be admissible). Suppose that we are given a certain direction at p; this may be specified by laying a smooth curve through p in the given direction, and describing the curve by a parameter t:

$$x^i = x^i(t) \equiv x_0^i + \lambda^i t + O(t^2) \tag{5}$$

for small t. Here x_0^i are the coordinates of p and the λ^i are constants which do not all vanish, provided that t is suitably chosen. By differentiating (5) we obtain

$$\left[\frac{dx^i}{dt}\right]_{t=0} = \lambda^i,$$

and these numbers λ^i define the given direction completely.† For example, given an analytic function f defined at p: $f(x)$, its derivative in the direction given by (5) is

$$\left[\frac{d}{dt} f(x(t))\right]_{t=0} = \lambda^i \left[\frac{\partial f}{\partial x^i}\right]_p . \tag{6}$$

† However, the direction depends only on the ratios of the λ's and not on the λ's themselves.

On the right the summation convention has been used, which consists in summing over the appropriate range—usually from 1 to n—with respect to any suffix (in this case i) which occurs twice. We shall use this convention throughout the text, except when otherwise stated. The subscript p in (6) indicates that the function in square brackets is to be evaluated at the point p.

The formula (6) suggests considering $L=\lambda^i.\partial/\partial x^i$ (evaluated at p) as an operator on $\mathscr{A}_p$ with real values. Accordingly we define: An operator of the form $L=\lambda^i.\partial/\partial x^i$, where the λ^i are any real constants, will be called a *tangent vector* at p. The definition involves a particular chart and we therefore give an alternative characterization of tangent vectors in

THEOREM 1.4.1. *A mapping L of $\mathscr{A}_p$ into R is a tangent vector if and only if it is linear over R:*

$$L(\alpha f+\beta g)=\alpha.Lf+\beta.Lg \quad (f,g\in\mathscr{A}_p;\ \alpha,\beta\in R), \tag{7}$$

and satisfies

$$L(fg)=Lf.g(p)+f(p).Lg \quad (f,g\in\mathscr{A}_p). \tag{8}$$

For clearly every tangent vector satisfies (7) and (8); equation (8) is just the product rule for differentiation. Now let L be a mapping of $\mathscr{A}_p$ into R which satisfies (7) and (8). Then for any constant function c in $\mathscr{A}_p$ we have

$$Lc=c.L1=c(L1.1+1.L1)=2Lc,$$

whence $Lc=0$. Now let $f\in\mathscr{A}_p$; near p we may express f as

$$f(x)=f(x_0)+c_i(x^i-x_0^i)+(x^i-x_0^i)(x^j-x_0^j)\,g_{ij}(x),$$

where $g_{ij}\in\mathscr{A}_p$, x_0^i are the coordinates of p and $c_i=[\partial f/\partial x^i]_p$. Applying L and using (7), we obtain

$$Lf=Lf(x_0)+c_i.L(x^i-x_0^i)+L((x^i-x_0^i)(x^j-x_0^j)\,g_{ij}(x)).$$

The first term vanishes because $f(x_0)$ is constant. For the last sum on the right we have, by (8),

$$\begin{aligned}&L((x^i-x_0^i)(x^j-x_0^j)\,g_{ij}(x))\\ &\quad=L(x^i-x_0^i)\,[(x^j-x_0^j)\,g_{ij}]_p+L(x^j-x_0^j)\,[(x^i-x_0^i)\,g_{ij}]_p\\ &\qquad\qquad+L(g_{ij})\,[(x^i-x_0^i)(x^j-x_0^j)]_p=0.\end{aligned}$$

Hence

$$Lf = Lx^i.c_i = Lx^i \left[\frac{\partial f}{\partial x^i}\right]_p, \quad \text{i.e.} \quad L = \lambda^i \frac{\partial}{\partial x^i}, \quad \text{where} \quad \lambda^i = Lx^i.$$

Thus L is a tangent vector, as was to be proved.

We have proved incidentally that a tangent vector L may, in any chart (x), be expressed by the formula

$$L = Lx^i \frac{\partial}{\partial x^i}. \tag{9}$$

If L_1 and L_2 are tangent vectors at p, and $\alpha_1, \alpha_2 \in R$, then $\alpha_1 L_1 + \alpha_2 L_2$, defined by $(\alpha_1 L_1 + \alpha_2 L_2)f = \alpha_1 L_1 f + \alpha_2 L_2 f$, is again a tangent vector at p; therefore the tangent vectors at p form a vector space over R. We shall denote this space by $\mathfrak{L}_p$.

THEOREM 1.4.2. *If (x) is an admissible chart at p, then the tangent vectors $\partial/\partial x^i$ form a basis of the space $\mathfrak{L}_p$.*

Proof. It is clear that the operator $\partial/\partial x^i : f \to [\partial f/\partial x^i]_p$ is in $\mathfrak{L}_p$, and equation (9) shows that the $\partial/\partial x^i$ span $\mathfrak{L}_p$. To prove their independence, suppose that there is a linear relation between them, say

$$L \equiv \lambda^i \frac{\partial}{\partial x^i} = 0. \tag{10}$$

Since the jth coordinate x^j is in $\mathscr{A}_p$, we may apply L to it:

$$0 = Lx^j = \lambda^i \frac{\partial x^j}{\partial x^i} = \lambda^j.$$

This shows that all the coefficients λ^j in (10) must be zero and the theorem follows.

Suppose now that (x) and (y) are two charts at p, of dimensions m and n respectively. By Theorem 1.4.2, each of the sets $\partial/\partial x^i$ $(i = 1, \ldots, m)$, $\partial/\partial y^j$ $(j = 1, \ldots, n)$ is a basis of $\mathfrak{L}_p$, and hence $m = n$. This proves

THEOREM 1.4.3. *All admissible charts at a given point of an analytic manifold have the same dimension.*

We may therefore define the *dimension* of an analytic manifold at a point p as the dimension of any chart at p. If a manifold has the same dimension n at all its points, it is said to be of dimension n. It is usual to require that a manifold shall have the same dimension at all its points; although we do not make this assumption explicitly, it is in fact true in all the cases which we consider.

As another application of (9) we derive the formula for a change of coordinates in $\mathfrak{L}_p$: If (y) is a second chart at p, then by (9),

$$\frac{\partial}{\partial y^k} = \left[\frac{\partial x^i}{\partial y^k}\right]_p \frac{\partial}{\partial x^i}.$$

Hence
$$\frac{\partial}{\partial y^k} = \alpha^i_k \frac{\partial}{\partial x^i}, \tag{11}$$

where $\alpha^i_k = [\partial x^i/\partial y^k]_p$ is the Jacobian matrix of the equations of transformation. Thus in the space $\mathfrak{L}_p$ of tangent vectors we can describe all coordinate changes by *linear* transformations.

Together with $\mathfrak{L}_p$ we wish to consider another vector space associated with the point p, namely, the dual space $\mathfrak{L}^*_p$ of $\mathfrak{L}_p$. For the sake of clarity we shall, in the next section, briefly review the properties of the dual space which we require.

1.5. The dual vector space. Let V be a vector space over R. *A linear form* on V is a mapping ξ of V into R: $v \to \langle v, \xi \rangle$, such that
$$\langle \alpha u + \beta v, \xi \rangle = \alpha \langle u, \xi \rangle + \beta \langle v, \xi \rangle \quad (u, v \in V;\ \alpha, \beta \in R).$$

The set of all linear forms on V will be denoted by V^*. We can define addition and multiplication by scalars in V^* as follows:

$$\langle u, \alpha\xi + \beta\eta \rangle = \alpha \langle u, \xi \rangle + \beta \langle u, \eta \rangle \quad (u \in V;\ \xi, \eta \in V^*;\ \alpha, \beta \in R).$$

It is easily seen that with these definitions V^* is a vector space over R. It is called the *dual space* of V, and $\langle u, \xi \rangle$ is called the *inner product* of u and ξ.

If $v_1, \ldots, v_n$ is a basis of V, then the general element v of V has the form
$$v = \lambda^i v_i, \tag{12}$$

and for any suffix j the mapping $v \to \lambda^j$ is a linear form on V, which thus defines an element ξ^j of V^*. With this notation the expression (12) for the general element of V becomes

$$v = \langle v, \xi^i \rangle v_i. \tag{13}$$

We shall prove that the ξ^i $(i = 1, \ldots, n)$ form a basis of V^*. For this purpose we note first that

$$\langle v_i, \xi^j \rangle = \delta^j_i, \tag{14}$$