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An Introduction to Homogenization

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Preface

The aim of homogenization theory is to establish the macroscopic behaviour of a system which is 'microscopically' heterogeneous, in order to describe some characteristics of the heterogeneous medium (for instance, its thermal or electrical conductivity). This means that the heterogeneous material is replaced by a homogeneous fictitious one (the 'homogenized' material), whose global (or overall) characteristics are a good approximation of the initial ones. From the mathematical point of view, this signifies mainly that the solutions of a boundary value problem, depending on a small parameter, converge to the solution of a limit boundary value problem which is explicitly described.

During the last ten years, we have both had the opportunity to give courses on homogenization theory for graduate and postgraduate students in several universities and schools of engineers. We realized that, while at the research level many excellent books have been written in the past, for the graduate level there was a lack of elementary reference books which could be used as an introduction to the field. Also, many classical and known results in linear homogenization, though currently taught, are not really available in the literature, either in books or in research articles. This lack naturally led to the idea to extend the material of our courses into the book we present here.

When teaching, we had to take into account that often the audience was not really familiar with the variational approach of partial differential equations (PDEs), which is the natural framework for homogenization theory. This is why we started the book with this topic. It is the subject of the first four chapters.

We have deliberately chosen not to present too many results, but to have those included all well explained. We focus our attention on the periodic homogenization of linear partial differential equations. A periodic distribution of the heterogeneities is a realistic assumption for a large class of applications. From the mathematical point of view, it contains the main difficulties arising in the study of composite materials.

Chapter 1 deals with two notions of convergence, the weak and the weak* one. This allows us to describe, in Chapter 2, the asymptotic behaviour of rapidly oscillating periodic functions.

In Chapter 3 we introduce the distributions and give the basic notions and theorems of Sobolev spaces. We pay particular attention to Sobolev spaces of periodic functions. The results of this chapter, as well as those of Chapter 1, are classical and are the necessary prerequisites for the variational approach of PDEs. We do not give their proofs but detailed references are quoted.

In Chapter 4 the variational approach to classical second order linear elliptic equations is introduced. Existence and uniqueness results for solutions of these equations with various boundary conditions are proved. Again we treat in detail the case of periodic boundary conditions.

From Chapter 5 to Chapter 12 we treat the periodic homogenization of several kinds of second order boundary value problems with rapidly oscillating periodic coefficients. We are concerned with elliptic equations, the linearized system of elasticity, the heat and the wave equations.

The model case is the Dirichlet problem for elliptic equations. The results concerning this case are the object of Chapters 5 and 6. In Chapter 5 we formulate the problem and list some physical examples. We also study two particular cases: the one-dimensional case and the case of layered materials. In Chapter 6 we state the general homogenization result and prove some properties of the homogenized coefficients.

The main homogenization methods for proving the general result are presented in Chapters 7–9. Thus, the multiple-scale method is described in Chapter 7. Chapter 8 is devoted to the oscillating test functions method. Finally, in Chapter 9 we introduce the two-scale convergence method.

In Chapter 8 we also prove some important related results, as for instance the convergence of energies and the existence of correctors. The convergence of eigenvalues and eigenvectors is also proved.

Chapters 10, 11 and 12 are devoted to the linearized system of elasticity, the heat equation and the wave equations respectively. In each chapter, we start by proving the existence and uniqueness of a solution. Then, we study the homogenization of the problem.

We conclude this book with a short overview of some general approaches to the study of the non-periodic case.

The idea of writing this book was to provide detailed proofs and tools adapted to the level we have in mind. Our hope is to give a background of homogenization theory not only to students, but also to researchers —in mathematics as well as in engineering, mechanics, or physics— who are interested in a mathematical introduction to the field.

Special thanks go to three of our colleagues. We thank Petru Mironescu for many helpful suggestions concerning the first four chapters. We also express our gratitude to Olivier Alvarez for his accurate reading of the manuscript and for his useful remarks and suggestions. Finally, we thank Thomas Lanchand-Robert for his valuable and patient help while we were typing this book in TeX.

This book represents for us the ultimate 'joint venture', which would have never been possible without a truly deep friendship and mutual understanding.

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Introduction

The aim of this book is to present the mathematical theory of the homogenization. This theory has been introduced in order to describe the behaviour of composite materials.

Composite materials are characterized by the fact that they contain two or more finely mixed constituents. They are widely used nowadays in industry, due to their properties. Indeed, they have in general a 'better' behaviour than the average behaviour of their individual constituents. Well-known examples are the superconducting multifilamentary composites which are used in the composition of optical fibres.

Generally speaking, in a composite the heterogeneities are small compared to its global dimension. So, two scales characterize the material, the microscopic one, describing the heterogeneities, and the macroscopic one, describing the global behaviour of the composite. From the macroscopic point of view, the composite looks like a 'homogeneous' material. The aim of 'homogenization' is precisely to give the macroscopic properties of the composite by taking into account the properties of the microscopic structure.

As a model case, let us fix our attention on the problem of the steady heat conduction in an isotropic composite.

Consider first a homogeneous body occupying Ω with thermal conductivity γ. For simplicity, we assume that the material is isotropic, which means that γ is a scalar. Suppose that f represents the heat source and g the temperature on the surface $\partial\Omega$ of the body, which we can assume to be equal to zero.

Then the temperature $u = u(x)$ at the point $x \in \Omega$ satisfies the following homogeneous Dirichlet problem:

$$\begin{cases} -\text{div}\ (\gamma \nabla u(x)) = f(x) & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where ∇u denotes the gradient of u defined by

$$\nabla u = \text{grad}\, u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N} \right).$$

Since γ is constant, this can be rewritten in the form

$$\begin{cases} -\gamma\, \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.2}$$

where $\Delta u = \text{div}(\text{grad}\ u)$. The flux of the temperature is defined by

$$q = \gamma \quad \text{grad}\ u. \tag{0.3}$$

This is a classical elliptic boundary value problem and it is well known that if f is sufficiently smooth, it admits a unique solution u which is twice differentiable and solves system (0.2) at any point x in Ω.

If now we consider a heterogeneous material occupying Ω, then the thermal conductivity takes different values in each component of the composite. Hence, γ is now a function, which is discontinuous in Ω, since it jumps over surfaces which separate the constituents. To simplify, suppose we are in presence of a mixture of two materials, one occupying the subdomain Ω_1 and the second one the subdomain Ω_2, with $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega = \Omega_1 \cup \Omega_2 \cup (\partial\Omega_1 \cap \partial\Omega_2)$.

Suppose also that the thermal conductivity of the body occupying Ω_1 is γ_1 and that of the body occupying Ω_2 is γ_2, i.e.

$$\gamma(x) = \begin{cases} \gamma_1 & \text{if } x \in \Omega_1 \\ \gamma_2 & \text{if } x \in \Omega_2. \end{cases}$$

Then the temperature and flux of the temperature in a point $x \in \Omega$ of the composite take respectively, the values

$$u(x) = \begin{cases} u_1(x) & \text{if } x \in \Omega_1 \\ u_2(x) & \text{if } x \in \Omega_2 \end{cases}$$

and

$$q = \begin{cases} q_1 = \gamma_1 \ \text{grad } u_1 & \text{in } \Omega_1 \\ q_2 = \gamma_2 \ \text{grad } u_2 & \text{in } \Omega_2. \end{cases}$$

The usual physical assumptions are the continuity of the temperature u and of the flux q at the interface of the two materials, i.e.

$$\begin{cases} u_1 = u_2 & \text{on } \partial\Omega_1 \cap \partial\Omega_2 \\ q_1 \cdot n_1 = q_2 \cdot n_2 & \text{on } \partial\Omega_1 \cap \partial\Omega_2, \end{cases} \tag{0.4}$$

where n_i is the outward normal unit vector to $\partial\Omega_i$, $i = 1, 2$ and $n_1 = -n_2$ on $\partial\Omega_1 \cap \partial\Omega_2$. Therefore, the temperature u is solution of the stationary thermal problem. Then the corresponding system (0.1) reads

$$\begin{cases} -\text{div } (\gamma(x) \text{ grad } u(x)) = f(x) & \text{in } \Omega_1 \cup \Omega_2 \\ u = 0 \quad \text{on } \partial\Omega \\ u_1 = u_2 \quad \text{on } \partial\Omega_1 \cap \partial\Omega_2 \\ q_1 \cdot n_1 = q_2 \cdot n_2 \quad \text{on } \partial\Omega_1 \cap \partial\Omega_2. \end{cases} \tag{0.5}$$

Formally, we can write this system in the form

$$\begin{cases} -\text{div } (\gamma(x) \text{ grad } u(x)) = f(x) & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{0.6}$$

Observe that from (0.4), it follows that the gradient of u is discontinuous. Moreover, in general, the flux q is not differentiable.

Taking into account these discontinuities, the question is what is the appropriate mathematical formulation of this problem and in which functional space can one have a solution (since one can not expect to have solutions of class C^1)?

An answer to these questions can be given by introducing a weak notion of solution. It is built on the notion of weak derivative, the so-called derivative in the sense of distributions. This is defined in Chapter 3, where we also introduce the Sobolev spaces which constitute the natural functional framework for weak solutions.

In the definition of a weak solution, problem (0.6) (or (0.5)) is replaced by a variational formulation, namely

$$\begin{cases} \text{Find } \ u \in H \text{ such that} \\ \displaystyle\sum_{i=1}^{N} \int_\Omega \gamma(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx = \int_\Omega f\, v\, dx, \quad \forall v \in H, \end{cases} \tag{0.7}$$

where H is an appropriate Sobolev space taking into account the boundary conditions on u. In (0.7) the derivatives are taken in the sense of distributions.

Of course, if u were sufficiently smooth, (0.7) and (0.6) would be equivalent. As seen above, this is not the case for a composite material, so the sense to be given to (0.6) is only that u solves (0.7). Let us point out that the equation in (0.7) is checked for any v belonging to the space H. This is why v is usually called a test function.

Existence and uniqueness results of a weak solution of (0.7) are proved in Chapter 4, where we also treat other kinds of boundary value problems.

Let us turn back to the question of the macroscopic behaviour of the composite material occupying Ω. Suppose that the heterogeneities are very small with respect to the size of Ω and that they are evenly distributed. This is a realistic assumption for a large class of applications.

From the mathematical point of view, one can modelize this distribution by supposing that it is a periodic one (see Fig. 0.1).

This periodicity can be represented by a small parameter, 'ε'.

Then the coefficient γ in (0.7) depends on ε and (0.7) reads

$$\begin{cases} \text{Find } \ u^\varepsilon \in H \ \text{ such that} \\ \displaystyle\sum_{i=1}^{N} \int_\Omega \gamma^\varepsilon(x) \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx = \int_\Omega f\, v\, dx. \qquad \forall v \in H. \end{cases} \tag{0.8}$$

A natural way to introduce the periodicity of γ^ε in (0.8) is to suppose that it has the form

$$\gamma^\varepsilon(x) = \gamma\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \mathbb{R}^N, \tag{0.9}$$

where γ is a given periodic function of period Y. This means that we are given a reference period Y, in which the reference heterogeneities are given. By definition (0.9), the heterogeneities in Ω are periodic of period εY and their size is

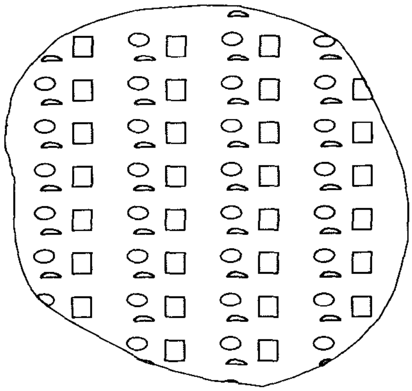

Fig. 0.1

of order of ε. Problem (0.8) is then written as follows:

$$\begin{cases} \text{Find } u^\varepsilon \in H \text{ such that} \\ \displaystyle\sum_{i=1}^{N} \int_\Omega \gamma\Big(\frac{x}{\varepsilon}\Big) \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx = \int_\Omega f\, v\, dx, \qquad \forall v \in H. \end{cases} \tag{0.10}$$

and Fig. 0.2 shows the periodic structure of Ω. Observe that two scales characterize our model problem (0.10), the macroscopic scale x and the microscopic one $\dfrac{x}{\varepsilon}$, describing the micro-oscillations.

The discontinuities of this problem make the model very difficult to treat, in particular from the numerical point of view. Also, the pointwise knowledge of the characteristic of the material does not provide in a simple way any information on its global behaviour.

Observe also that making the heterogeneities smaller and smaller means that we 'homogenize' the mixture and from the mathematical point of view this means that ε tends to zero. Taking $\varepsilon \to 0$ is the mathematical 'homogenization' of problem (0.10).

Many natural questions arise:

(1) Does the temperature u^ε converge to some limit function u^0?

(2) If that is true, does u^0 solve some limit boundary value problem?

(3) Are then the coefficients of the limit problem constant?

(4) Finally, is u^0 a good approximation of u^ε?

Answering these questions is the aim of the mathematical theory of 'homogenization'.

These questions are very important in the applications since, if one can give positive answers, then the limit coefficients, as it is well known from engineers

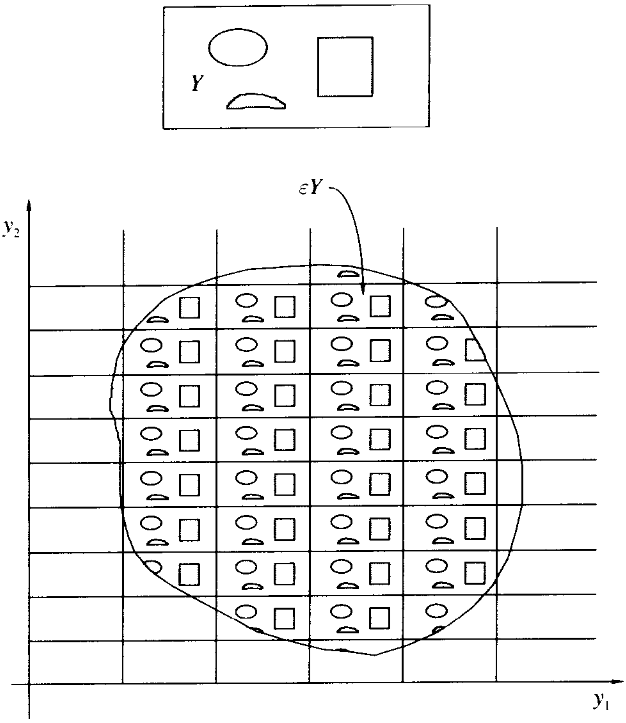

Fig. 0.2

and physicians, are good approximations of the global characteristics of the composite material, when regarded as an homogeneous one. Moreover, replacing the problem by the limit one allows us to make easy numerical computations.

The first remark is that the function γ^ε converges in a weak sense to the mean value of γ , i.e. one has

$$\int_\Omega \gamma^\varepsilon(x)\, v(x)\, dx \longrightarrow \int_\Omega \mathcal{M}_Y(\gamma)\, v(x)\, dx, \tag{0.11}$$

for any integrable function v. Here the mean value $\mathcal{M}_Y(\gamma)$ is defined by

$$\mathcal{M}_Y(\gamma) = \frac{1}{|Y|}\int_Y \gamma(y)\, dy.$$

This result on the convergence of periodic functions is proved in Chapter 2. The notion of weak convergence and related properties are presented in Chapter 1.

One can also (thanks to weak-compactness results stated in Chapter 1) show that u^ε converges to some function u^0 and that ∇u^ε weakly converges to ∇u^0.

The question is whether these convergences and convergence (0.11) are sufficient to homogenize problem (0.10). To do that, one has to pass to the limit in the product $\gamma^\varepsilon \nabla u^\varepsilon$. This is the main difficulty in homogenization theory. Actually, in general (see Chapters 1 and 2), the product of two weakly convergent sequences does not converge to the product of the weak limits. In Section 5.1 we show that there is a vector function ξ, weak limit of the product $\gamma^\varepsilon \nabla u^\varepsilon$ and satisfying the equation

$$-\text{div } \xi = f. \tag{0.12}$$

But

$$\xi \neq \mathcal{M}_Y(\gamma)\nabla u^0,$$

so that from (0.12) one cannot easily deduce an equation satisfied by u^0. This already occurs in the one-dimensional case where Ω is some interval $]d_1, d_2[$. One has (see Section 5.3)

$$\xi = \frac{1}{\mathcal{M}_Y\left(\frac{1}{\gamma}\right)} \frac{du^0}{dx}.$$

Moreover, u^0 is the unique solution of the homogenized problem

$$\begin{cases} -\dfrac{d}{dx}\left(\dfrac{1}{\mathcal{M}_Y\left(\frac{1}{\gamma}\right)} \dfrac{du^0}{dx}\right) = f \quad \text{in }]d_1, d_2[\\ u^0(d_1) = u^0(d_2) = 0. \end{cases}$$

Clearly, $\xi \neq \mathcal{M}_Y(\gamma)\nabla u^0$, since

$$\frac{1}{\mathcal{M}_Y\left(\frac{1}{\gamma}\right)} \neq \mathcal{M}_Y(\gamma).$$

Even for the one-dimensional case this homogenization result is not trivial. The situation is of course, more complicated in the general N-dimensional case. The one-dimensional result could suggest that in the N-dimensional case the limit problem can be described in terms of the mean value of γ^{-1}. This is not true, as can already be seen in the case of layered materials studied in Section 5.4, where γ depends only on one variable, say x_1. In this case, the homogenized problem of (0.10) is

$$\begin{cases} -\text{div } (A^0 \nabla u^0) = f \quad \text{in } \Omega \\ u^0 = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{0.13}$$

where the homogenized matrix A^0 is constant, diagonal and given by

$$A^0 = \begin{pmatrix} \dfrac{1}{\mathcal{M}_Y(\gamma^{-1})} & 0 & \dots & 0 \\ 0 & \mathcal{M}_Y(\gamma) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathcal{M}_Y(\gamma) \end{pmatrix}.$$

Let us point out that the homogenized material is no longer isotropic, since A^0 is not of the form $a^0 I$.

Observe also that in these particular examples of the one-dimensional case and of layered materials, the homogenized coefficients are algebraic formulas involving γ.

For the general N-dimensional case, as seen in Chapter 6, the homogenized problem is still of the form (0.13). The coefficients of A^0 are defined by means of some periodic functions which are the solutions of some boundary value problems of the same type as (0.10) posed in the reference cell Y. The coefficients a_{ij}^0 of the matrix A^0 are defined by

$$a_{ij}^0 = \frac{1}{|Y|}\int_Y \gamma\, \delta_{ij}\, dy - \frac{1}{|Y|}\int_Y \gamma \frac{\partial \chi_j}{\partial y_i}\, dy, \quad \forall\, i,j = 1,\dots,N, \tag{0.14}$$

where δ_{ij} is the Kronecker symbol. The function χ_j for $j = 1,\dots,N$ is the solution of the problem

$$\begin{cases} -\mathrm{div}\, (\gamma(y)\nabla \chi_j) = -\dfrac{\partial \gamma}{\partial y_j} & \text{in } Y \\ \chi_j \quad Y\text{-periodic} \\ \mathcal{M}_Y(\chi_j) = 0. \end{cases} \tag{0.15}$$

This result can be proved by different methods. We present in this book three of them.

In Chapter 7 we use the multiple-scale method, which consists of searching for u^ε in the form

$$u^\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots, \tag{0.16}$$

where $u_j = u_j(x, y)$ are Y-periodic in the second variable y.

The multiple-scale method is a classical one, widely used in mechanics and physics for problems containing several small parameters describing different scalings. It is well adapted to the periodic framework in which we work in this book. Its interest is that in general, it permits us to obtain formally the homogenized problem.

Chapter 8 is devoted to the oscillating test functions method introduced by L. Tartar. As we have seen above, in problem (0.10) the function u^ε is continuous at the interface $\partial\Omega_1 \cap \partial\Omega_2$ but its gradient is not, and behaves in such a way that the flux $\gamma\nabla u^\varepsilon$ remains continuous. The idea of Tartar's method is to construct test functions $v = w_j^\varepsilon \varphi$ for (0.10) having the same kind of discontinuities as u^ε and having a known limit. For our example, one has

$$w_j^\varepsilon(x) = -\chi_j\left(\frac{x}{\varepsilon}\right) + x_j, \quad j = 1,\dots,N, \tag{0.17}$$

and φ is a smooth function vanishing on $\partial\Omega$. Using these test functions in the variational formulation (0.10), one is able to pass to the limit and identify ξ in terms of u^0. Actually, one obtains ξ in the form

$$\xi = A^0 \nabla u^0.$$

This together with (0.12) gives the homogenized problem.

In Chapter 8 we also prove a corrector result which for the model problem (0.10) is the following. Let us introduce the (corrector) matrix $C^\varepsilon = (C^\varepsilon_{ij})_{1\le i,j\le N}$ defined by

$$C^\varepsilon_{ij}(x) = \frac{\partial w_j}{\partial y_i}\left(\frac{x}{\varepsilon}\right),$$

where w_j is given by (0.17). Then,

$$\nabla u^\varepsilon - C^\varepsilon \nabla u^0 \longrightarrow 0$$

in a usual (strong) convergence.

Moreover, let us observe that, when applying the multiple-scale method one finds

$$u_1(x,y) = -\sum_{j=1}^{N} \chi_j(y)\frac{\partial u_0}{\partial x_j}.$$

Therefore

$$\begin{aligned}\nabla u^\varepsilon(x) &= \nabla u^0(x) - \sum_{k=1}^{N} \nabla_y \chi_k\left(\frac{x}{\varepsilon}\right)\frac{\partial u^0}{\partial x_k}(x) - \varepsilon\sum_{k=1}^{N} \widehat{\chi}_k\left(\frac{x}{\varepsilon}\right)\nabla\left(\frac{\partial u^0}{\partial x_k}\right)(x) + \cdots\\ &= C^\varepsilon(x)\nabla u^0(x) - \varepsilon\sum_{k=1}^{N} \chi_k\left(\frac{x}{\varepsilon}\right)\nabla\left(\frac{\partial u^0}{\partial x_k}\right)(x) + \cdots.\end{aligned}$$

Hence $C^\varepsilon(x)\nabla u^0(x)$ is the first term in the asymptotic expansion (0.16) of ∇u^ε.

In the same chapter we also give further properties of the homogenized problem.

In Chapter 9 we prove again the convergence result by the two-scale method which takes into account the two scales of the problem and introduces the notion of 'two-scale convergence'. This convergence is tested on functions of the form $\psi(x, x/\varepsilon)$. One of the interests of the two-scale method is that it justifies mathematically the formal asymptotic development (0.16).

In Chapters 10, 11, and 12 we treat respectively the linearized system of elasticity, the heat equation and the wave equation. For each problem, we first prove the existence and uniqueness of the solution, then we study their homogenization.

Finally, Chapter 13 contains a short overview of some methods used in the general non-periodic case. In particular, we fix our attention on G-convergence and H-convergence.

1

Weak and weak* convergences in Banach spaces

We recall in this chapter the main properties of weak and weak* convergence in a Banach space. We also detail these notions for the particular case of L^p-spaces.

Let us begin by recalling the notions of a Banach and a Hilbert space which are the functional spaces in which we work in this book. The spaces we consider in this book are all real.

Definition 1.1. A mapping

$$\|\cdot\| : x \in E \longmapsto \|x\| \in \mathbb{R}_+$$

is called a *norm* on the vector space E iff

$$\begin{cases} \|x\| = 0 \Longleftrightarrow x = 0 \\ \|\lambda x\| = |\lambda|\|x\|, \quad \text{for any } \lambda \in \mathbb{R},\ x \in E \\ \|x + y\| \leq \|x\| + \|y\|, \quad \text{for any } x,\ y \in E. \end{cases}$$

Then E is called a *normed space* and its norm is denoted by $\|\cdot\|_E$.

Moreover, E is called a Banach space iff it is complete with respect to the following convergence (called strong convergence):

$$x_n \to x \text{ in } E \Longleftrightarrow \|x_n - x\|_E \to 0.$$

Definition 1.2. Let H be a real linear space. A mapping

$$(\cdot,\cdot)_H : (x,y) \in H \times H \longmapsto (x,y)_H \in \mathbb{R}$$

is called a *(real) scalar product* iff

$$\begin{cases} (x,x)_H > 0 \Longleftrightarrow x \neq 0, \\ (x,y)_H = (y,x)_H, \quad \text{for any } x,\ y \in H \\ (\lambda x + \mu y, z)_H = \lambda(x,z)_H + \mu(y,z)_H, \quad \text{for any } \lambda,\ \mu \in \mathbb{R},\ x,\ y,\ z \in H. \end{cases}$$

Moreover, if H is a Banach space with respect to the norm associated to this scalar product, i.e. with

$$\|x\|_H = (x,x)_H^{\frac{1}{2}},$$

then H is called a Hilbert space.