

# CONTEMPORARY MATHEMATICS

684

## Categorification in Geometry, Topology, and Physics

Anna Beliakova  
Aaron D. Lauda  
Editors



American Mathematical Society

The emergent mathematical philosophy of categorification is reshaping our view of modern mathematics by uncovering a hidden layer of structure in mathematics, revealing richer and more robust structures capable of describing more complex phenomena. Categorification is a powerful tool for relating various branches of mathematics and exploiting the commonalities between fields. It provides a language emphasizing essential features and allowing precise relationships between vastly different fields.

This volume focuses on the role categorification plays in geometry, topology, and physics. These articles illustrate many important trends for the field including geometric representation theory, homotopical methods in link homology, interactions between higher representation theory and gauge theory, and double affine Hecke algebra approaches to link homology.

The companion volume (Contemporary Mathematics, Volume 683) is devoted to categorification and higher representation theory.

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# Categorification in Geometry, Topology, and Physics

Dedicated to Christian Blanchet  
on the occasion of his sixtieth birthday

## Preface

The emergent mathematical philosophy of categorification is reshaping our view of modern mathematics by uncovering a hidden layer of structure in mathematics, revealing richer and more robust structures capable of describing more complex phenomena. This philosophy has led to a number of shocking new results and has ushered in a new area of interaction between algebra, geometry, and theoretical physics. Categorification is a powerful tool for relating various branches of mathematics and exploiting the commonalities between fields. It provides a language emphasizing essential features and allowing precise relationships between vastly different fields.

Categorification exemplifies the duality between pure mathematics and theoretical physics, with ideas from one area inspiring innovation in the other. Key concepts are exchanged between fields and transported to other areas using the powerful descriptive ability enabled by categorification. Indeed, the term “categorification” originated in work of Crane and Frenkel who were studying techniques for extending 3-dimensional topological quantum field theories to 4-dimensions. This area has rapidly developed over the last twenty years far beyond these original considerations, blossoming into an exciting and dynamic area of modern mathematics, with deep relations to geometric representation theory, low-dimensional topology, algebraic geometry, and mathematical physics.

Classical structures in representation theory, such as simple Lie algebras and their representations, allow deformations into quantum groups and modules over them, as discovered by Drinfeld and Jimbo. In turn, quantum groups lead to a representation-theoretical interpretation of the Jones, Kauffman, HOMFLY-PT and other link polynomials. These theories are closely connected to physics via the Chern-Simons path integral, as shown by Witten in his seminal paper on the Jones polynomial. More recently, it became clear that quantum deformation was only a prelude into deeper structural beauty and meaning. In the work of Khovanov, Ozsvath, Szabo, Rasmussen, Rozansky, and many others, *quantum link invariants* were interpreted as Euler characteristics of various bigraded and multi-graded link homology theories. These theories carry four-dimensional information, providing invariants of link cobordisms. The resulting link homology theories appear to be a nexus between some of the most sophisticated directions in modern mathematics and theoretical physics.

The discovery of link homology gave rise to new advances in representation theory, as quantum groups and related algebras were categorified. The resulting higher representation theory created a new bridge between representation theory and the link homology theories, which despite the increasing number of publications in this area is still far from being fully explored.

This volume was inspired by the conference Categorification in Algebra, Geometry, and Physics (a conference in honor of the 60th Birthday of Christian Blanchet), which took place at IESC, Menasina, Cargese, France, from May 4th to May 8th, 2015. This conference made clear that categorification is a rapidly emerging area of intense study. It also elucidated the need for a reference for newcomers to the field to learn the types of tools used in categorification, the problems where these tools have been successfully applied, and the future directions in which the field is moving. Our aim with this volume is to address this need. To this end, we have solicited articles from experts in categorification from around the world who were invited to share their unique perspective.

This volume focuses on the role categorification plays in geometry, topology, and physics. These articles illustrate many important trends for the field including geometric representation theory, homotopical methods in link homology, interactions between higher representation theory and gauge theory, and DAHA approaches to link homology.

The organizers wish to thank the John Templeton Foundation and the University of Zurich for their generous support in making this conference possible.

Anna Beliakova  
Aaron Lauda

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## Geometry and categorification

Ben Webster

**ABSTRACT.** We describe a number of geometric contexts where categorification appears naturally: coherent sheaves, constructible sheaves and sheaves of modules over quantizations. In each case, we discuss how “index formulas” allow us to easily perform categorical calculations, and readily relate classical constructions of geometric representation theory to categorical ones.

### 1. Introduction

“Categorification” is a very flexible concept. It simply refers to the idea that it can be very interesting to take a set and add morphisms between its objects. Its very flexibility means that it is an idea which must be employed carefully. It has not proven very effective to start with a simple algebraic object and to hunt aimlessly for categorifications of it. It is much more reliable to have a “machine” which produces categories for you in a way that gives you some hope of understanding how they decategorify.

Thus, geometry is a natural context for categorification because it is a natural source of categories. The categories that appear in geometry also have a natural geometric toolkit for producing functors (using push-pull along correspondences) and calculation (using index formulas). Both of these can be more difficult to understand in other approaches to categorification, such as algebraic or diagrammatic. The focus of this paper will be on describing some of the basic ways of applying geometry to construct categories, how the underlying geometry can help with understanding these categories, and how to apply these principles in some of the most illuminating special cases.

Categorification can also help us to better understand geometry. The categories which appear naturally in this context shed light on the nature of the spaces they are connected to. Often, the full structure of a category like coherent sheaves on an algebraic variety is simply “too rich for our blood,” an amount of information that exceeds our ability to take it in. Decategorification allows us cut out much of the extraneous complication and understand some of the structure of this category, and thus learn something about the underlying space.

This paper is structured around 3 different geometric contexts which naturally lead to categorification:

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- In Section 2, we consider categories of coherent sheaves and algebraic K-theory. This is arguably the first place in the literature where the modern philosophy of categorification appears, and the most likely to be somewhat familiar to the general reader.
- In Section 3, we consider categories of constructible sheaves and the function-sheaf correspondence. While perhaps a more specialized taste, this is actually an incredibly powerful theory, with connections to deep number theory. In this author's opinion, it is one any aspiring categorifier should know a bit of.
- In Section 4, we consider categories of sheaves of modules over quantizations. This is the least familiar context, and one still under development. Unlike the other two examples, we have not had the benefit of having Grothendieck around to help us with it. However, progress on it has been made, which we will briefly discuss here.

## 2. K-theory

The first appearance of categorification in its modern form was in topological K-theory. Given a topological space  $X$ , we can consider the additive category of vector bundles on  $X$ . To better understand this category we consider its **Grothendieck group**  $K(X)$ . This is the abelian group generated by symbols  $[A]$  for  $A$  a vector bundle, subject to the relation

$$(1) \quad [A] + [B] = [A \oplus B].$$

Note, most authors consider vector bundles up to stable equivalence, which is the same as considering the kernel of the map  $K(X) \rightarrow \mathbb{Z}$  sending a vector bundle to its rank. Of course, this construction has many variations where we consider bundles with additional structure. See the classic books of Atiyah [Ati67] or Karoubi [Kar08] for more details on K-theory.

**2.1. Coherent sheaves.** This topological introduction is perhaps a little misleading. We'll instead be working with algebraic varieties. Both a curse and a blessing of this geometric approach is that almost every construction that appears has several variations that make sense in different contexts. Throughout, I'll usually work in whatever context is most convenient for me. Thus, I could consider holomorphic vector bundles on a complex manifold or locally free coherent sheaves on a scheme or variety (over the complex numbers or some other field). For a projective variety over  $\mathbb{C}$ , these notions are the same by [Ser56], so the reader will not lose much by thinking about whichever one they prefer.

Now, we will more seriously study the category  $\text{Coh}(X)$  of coherent sheaves on a scheme  $X$ . Readers who are less happy with algebraic geometry might also think about the category of modules over a commutative ring  $R$ , which is the same as  $\text{Coh}(\text{Spec } R)$ , the coherent sheaves on the spectrum  $\text{Spec } R$  of  $R$ .

**DEFINITION 2.1.** *Attached to this category, we have two natural Grothendieck groups:*

- Let  $K_0(X) := K_0(\text{Coh}(X))$  be the formal span of  $[\mathcal{F}]$  for all **locally free**<sup>1</sup> coherent sheaves  $\mathcal{F}$  on  $X$ , modulo the relation

$$(2) \quad [\mathcal{E}] - [\mathcal{F}] + [\mathcal{G}] = 0$$

---

<sup>1</sup>Those of you thinking about  $R$ -modules should restrict to projective  $R$ -modules.

for any short exact sequence

$$(3) \quad 0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0,$$

which is a slight modification of the relation (1), and

- Let  $G_0(X) := G_0(\text{Coh}(X))$  be the span of  $[\mathcal{F}]$  for all coherent sheaves modulo the same relation (2).

This modified relation is needed since neither of these categories is semi-simple, while every short exact sequence of topological vector bundles splits.

Note that there's an obvious homomorphism  $K_0(X) \rightarrow G_0(X)$ , but this need not be an isomorphism. If  $X$  is quasi-projective and smooth, then it induces a natural isomorphism  $K_0(X) \cong G_0(X)$  since every sheaf has a finite length locally free resolution<sup>2</sup>. On the other hand, this will not hold in many other cases.

**REMARK 2.2.** If  $R = \mathbb{C}[t]/(t^2)$ ,  $X = \text{Spec } R$ , then  $K_0(X)$  is spanned by the class of the regular module  $R$ , and  $G_0(X)$  is spanned by the class of the 1-dimensional module  $\mathbb{C} \cong R/tR \cong tR$ . The short exact sequence

$$0 \longrightarrow tR \longrightarrow R \longrightarrow R/tR \longrightarrow 0,$$

shows that under the isomorphism  $G_0(X) \cong \mathbb{Z}$ , the subgroup  $K_0(X)$  is sent to  $2\mathbb{Z}$ . The difference between this case and the smooth case is that the minimal projective resolution of  $R/tR$  is the infinite complex

$$\cdots \longrightarrow R \xrightarrow{t} R \xrightarrow{t} \cdots \xrightarrow{t} R \longrightarrow R/tR.$$

Note that  $K_0(X)$  is a ring, with multiplication given by  $[\mathcal{E}][\mathcal{F}] = [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}]$ . We cannot endow  $G_0(X)$  with a compatible ring structure in general since  $\otimes_{\mathcal{O}_X}$  is not exact. As we'll discuss below, we can sometimes fix non-exactness by considering higher derived functors, but our example above shows that this can't work here: if  $X = \text{Spec } \mathbb{C}[t]/(t^2)$  then  $K_0(X) \cong \mathbb{Z}$  as a ring, so  $G_0(X)$  can only be a ring if  $1/2\mathbb{Z}$  is. The problem is that  $\text{Tor}_R^i(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$  for all  $i$ , so we would have to use the divergent series  $1/2 = 1 - 1 + 1 - 1 + \cdots$ .

**2.2. Functoriality.** Our first task is to understand how maps between varieties induce functors between categories and thus maps between K-groups. Of course, the desired functors are pushforward and pullback of coherent sheaves, as defined in [Har77, II.5]. Unfortunately, interpreted naively, neither of these functors is exact. Pushforward  $f_*$  is left exact, and thus has right derived functors  $\mathbf{R}^i f_*$  (see [Har77, III.8]), and pullback  $f^*$  is right exact and thus has left derived functors  $\mathbf{L}^i f^*$ . These have the usual long exact sequences

$$(4) \quad \cdots \longrightarrow \mathbf{R}^{i-1} f_* \mathcal{G} \longrightarrow \mathbf{R}^i f_* \mathcal{E} \longrightarrow \mathbf{R}^i f_* \mathcal{F} \longrightarrow \mathbf{R}^i f_* \mathcal{G} \longrightarrow \mathbf{R}^i f_* \mathcal{E} \longrightarrow \mathbf{R}^{i+1} f_* \mathcal{E} \longrightarrow \cdots$$

$$(5) \quad \cdots \longrightarrow \mathbf{L}^{i+1} f^* \mathcal{G} \longrightarrow \mathbf{L}^i f^* \mathcal{E} \longrightarrow \mathbf{L}^i f^* \mathcal{F} \longrightarrow \mathbf{L}^i f^* \mathcal{G} \longrightarrow \mathbf{L}^i f^* \mathcal{E} \longrightarrow \mathbf{L}^{i-1} f^* \mathcal{E} \longrightarrow \cdots$$

which show that the maps

$$(6) \quad [\mathcal{F}] \mapsto \sum_{i=0}^{\infty} (-1)^i [\mathbf{R}^i f_* \mathcal{F}] \quad [\mathcal{G}] \mapsto \sum_{i=0}^{\infty} (-1)^i [\mathbf{L}^i f^* \mathcal{G}]$$

<sup>2</sup>In fact, by a theorem of Serre, a quasi-projective variety is smooth if and only if every coherent sheaf has a finite length resolution.

are compatible with the relation (2) whenever these infinite sums make sense. As often happens, the two different maps above make sense for the two different versions of the Grothendieck group:

**PROPOSITION 2.3.** *For a projective morphism  $f: X \rightarrow Y$ , the formulas of (6) define maps*

$$(7) \quad G_0(f_*): G_0(X) \rightarrow G_0(Y) \quad K_0(f^*): K_0(Y) \rightarrow K_0(X)$$

**PROOF.** For any coherent sheaf  $\mathcal{F}$  on  $X$ , the sheaves  $\mathbf{R}^i f_* \mathcal{F}$  are coherent by [Har77, III.8.8(b)], and vanish for  $i > \dim X$  by [Har77, III.2.7]. Thus, the first sum of (6) is finite and well-defined. Note that we could not do this for  $K_0$  since  $\mathbf{R}^i f_* \mathcal{F}$  might not be locally free even if  $\mathcal{F}$  is.

For a locally free sheaf  $\mathcal{G}$  on  $Y$ , the pullback  $f^* \mathcal{G}$  is locally free and the higher pullbacks  $L^i f^* \mathcal{G}$  for  $i > 0$  are 0, so the second sum of (6) is well-defined (and in fact only has 1 non-zero term).  $\square$

The map  $\text{Spec } \mathbb{C} \rightarrow \text{Spec } R$  induced by the unique ring homomorphism  $R \rightarrow \mathbb{C}$  illustrates that it's impossible to define these maps on the "wrong" Grothendieck group:

- the regular module  $\mathbb{C}$  is free, but its pushforward  $f_* \mathbb{C}$  is just the module  $R/tR$ , which doesn't have a corresponding class in  $K_0$ .
- the pullback  $L^i f^*(R/tR) \cong \mathbb{C}$  for all  $i$ , so the second sum in (6) doesn't converge.

Of course, this map is somewhat pathological, and there are conditions one can impose that will guarantee these maps make sense. In particular:

- If the source and target are smooth and quasi-projective, then we can freely switch between  $K_0$  and  $G_0$  and so  $G_0(f^*)$  and  $K_0(f_*)$  make sense for a projective (or more generally, proper) morphism.
- If  $f$  is flat, then  $f^*$  is exact, and  $G_0(f^*)$  is well-defined.

**2.3. Chern character and index formulas.** Thus, we can supply ourselves with a great number of categories and exciting functors between them. But as we said in the introduction, this is particularly powerful because we can understand the Grothendieck group and calculate the behavior of these maps in geometric terms. Assume from now on that  $X$  is smooth and projective over  $\mathbb{C}$ .

**THEOREM 2.4.** *There is a unique homomorphism  $\text{ch}: K_0(X) \cong G_0(X) \rightarrow H^*(X; \mathbb{Q})$  called **Chern character** which is:*

- (1) *compatible with pullback*

$$\text{ch}(f^* \mathcal{F}) = f^* \text{ch}(\mathcal{F})$$

- (2) *sends the class  $[L]$  of a line bundle  $L$  to*

$$\exp(c_1(L)) = 1 + c_1(L) + \frac{c_1(L)^2}{2} + \dots$$

where  $c_1(L)$  is the first Chern class, the cohomology class dual to the divisor defined by a meromorphic section of  $L$ .

This definition is well-defined because of the splitting principle: given a vector bundle  $E \rightarrow X$  of rank  $n$ , we can consider the flag space  $\text{Fl}(E)$  given by pairs of a

point  $x \in X$  and a complete flag  $V_1 \subset V_2 \subset \cdots \subset V_n = E_x$ . The cohomology ring is given by

$$H^*(\mathrm{Fl}(E); \mathbb{Q}) \cong H^*(X; \mathbb{Q})[\alpha_1, \dots, \alpha_n] / (e_k(\alpha) = c_k(E));$$

here the classes  $\alpha_i$  are the first Chern classes of the induced vector bundles  $V_i/V_{i-1}$ . We can use  $e_k(\alpha) = c_k(E)$  as a definition of the Chern classes, and  $\mathrm{ch}(E) = \sum_{i=1}^n e^{\alpha_i}$ . Note that this implicitly gives a complicated but concrete formula for  $\mathrm{ch}$  in terms of Chern classes, since each homogeneous part is a symmetric polynomial in Chern roots  $\alpha_i$ , and thus a polynomial in the Chern classes. We can also define the Todd class  $\mathrm{td}(E) = \prod_{i=1}^n \frac{\alpha_i}{1 - e^{-\alpha_i}}$ .

**REMARK 2.5.** *Note that even if  $X$  is defined over a field other than  $\mathbb{C}$ , we can still define this homomorphism, with the target given by the Chow ring  $A^*(X; \mathbb{Q})$ , the ring spanned by subvarieties of  $X$  modulo rational equivalence.*

The Chern character is certainly not an isomorphism, but is not so far from being one either:

**PROPOSITION 2.6.** *The kernel of  $\mathrm{ch} \otimes \mathbb{Q}: K_0(X) \otimes \mathbb{Q} \rightarrow H^*(X; \mathbb{Q})$  is spanned by elements  $[\mathcal{F}] - [\mathcal{G}]$  where we have an isomorphism of underlying topological  $\mathbb{C}$ -vector bundles  $\mathcal{F} \cong_{\mathbb{C}} \mathcal{G}$ .*

The inelegance of this theorem comes from the fact that we are relating two “incompatible” structures. If we consider the topological K-theory of  $X$  as a manifold, or replace  $H^*(X; \mathbb{Q})$  by the Chow ring  $A^*(X; \mathbb{Q})$ , then this map will be an isomorphism. For many nice varieties, we have an isomorphism  $H^*(X; \mathbb{Q}) \cong A^*(X; \mathbb{Q})$ , so the map above becomes an isomorphism. For example, this is the case for any variety which has an affine paving.

While there are many things to be said about K-theory, perhaps the most important for us is the first example of an index formula: the Grothendieck-Hirzebruch-Riemann-Roch theorem.

**THEOREM 2.7** (Grothendieck, Hirzebruch; [BS58, §7]). *Let  $X, Y$  be smooth and projective over  $\mathbb{C}$ . Then for any map  $f: X \rightarrow Y$  and any locally free sheaf  $\mathcal{F}$ , we have that*

$$\mathrm{ch}(f_*\mathcal{F}) \mathrm{td}(T_Y) = f_*(\mathrm{ch}(\mathcal{F}) \mathrm{td}(T_X)).$$

*In particular, in the case where  $Y = \mathrm{Spec} \mathbb{C}$  is a point, we have that*

$$\sum_{i=0}^{\infty} (-1)^i \dim H^i(X; \mathcal{F}) = \int_X \mathrm{ch}(\mathcal{F}) \mathrm{td}(T_X).$$

Since this is a particularly focus of categorifiers, let us note that using internal Hom allows us to compute the Euler form on  $K_0(X)$ , which is defined by

$$\langle [\mathcal{E}], [\mathcal{F}] \rangle = \sum_{i=0}^{\infty} \mathrm{Ext}^i(\mathcal{E}, \mathcal{F}).$$

This sum is well-defined because we have that

$$\mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) = H^i(X; \mathcal{E}^\vee \otimes \mathcal{F}),$$

which vanishes for  $i > \dim X$ . This also allows us to see that

$$\langle [\mathcal{E}], [\mathcal{F}] \rangle = \int_X \mathrm{ch}(\mathcal{F}) \mathrm{ch}(\mathcal{E}^\vee) \mathrm{td}(T_X).$$

**2.4. The Weyl character formula.** One of the most important categorification problems is understanding the characters of representations of groups. For complex simple Lie groups, this problem is solved by the Weyl character formula. While there are many proofs of this beautiful formula, one of the most remarkable is obtained by combining GHRR as above with the Borel-Weil-Bott theorem, which shows that for each highest weight  $\lambda$ , there is a line bundle  $\mathcal{L}_\lambda$  on the flag variety  $X := G/B$  of  $G$  such that  $H^0(X; \mathcal{L}_\lambda) \cong V_\lambda$  and  $H^i(X; \mathcal{L}_\lambda) = 0$  for  $i > 0$ . Thus, we have that:

$$\dim V_\lambda = \int_X \text{ch}(\mathcal{L}_\lambda) \text{td}(T_X).$$

Consider the symmetric algebra  $\text{Sym}^\bullet(\mathfrak{t}_\mathbb{Q}^*)$  in the weights of the maximal torus  $T$ , and let  $\epsilon(f)$  be the constant term of  $f \in \text{Sym}^\bullet(\mathfrak{t}_\mathbb{Q}^*)$ . We can identify  $H^*(X; \mathbb{Q})$  with the symmetric algebra  $\text{Sym}^\bullet(\mathfrak{t}_\mathbb{Q}^*)$  modulo  $\text{Sym}^\bullet(\mathfrak{t}_\mathbb{Q}^*)^W_+$ , the polynomials with  $\epsilon(f) = 0$  which are symmetric under the action of the Weyl group; in this realization, the operation of integration is given by

$$(8) \quad \int_X p = \epsilon \left( \frac{\sum_{w \in W} (-1)^w w \cdot p}{\prod_{\alpha \in \Delta^+} \alpha} \right).$$

The positive roots  $\alpha \in \Delta^+$  are the Chern roots of the tangent bundle, so we have  $\text{td}(T_X) = \prod_{\alpha \in \Delta^+} \frac{\alpha}{1 - e^{-\alpha}}$ , and the Chern character of the line bundle is  $\text{ch}(\mathcal{L}_\lambda) = e^\lambda$ , so

$$\text{ch}(\mathcal{L}_\lambda) \text{td}(T_X) = \frac{e^{\lambda + \rho} \prod_{\alpha \in \Delta^+} \alpha}{\prod_{\alpha \in \Delta^+} e^{\alpha/2} - e^{-\alpha/2}}.$$

Thus, we have that

$$(9) \quad \begin{aligned} \dim V_\lambda &= \int_X \text{ch}(\mathcal{L}_\lambda) \text{td}(T_X) \\ &= \epsilon \left( \frac{1}{\prod_{\alpha \in \Delta^+} \alpha} \sum_{w \in W} (-1)^w \frac{e^{w(\lambda + \rho)} \prod_{\alpha \in \Delta^+} \alpha}{\prod_{\alpha \in \Delta^+} e^{\alpha/2} - e^{-\alpha/2}} \right) \\ &= \epsilon \left( \frac{\sum_{w \in W} (-1)^w e^{w(\lambda + \rho)}}{\prod_{\alpha \in \Delta^+} e^{\alpha/2} - e^{-\alpha/2}} \right). \end{aligned}$$

If we evaluate the RHS using L'Hôpital's rule, we obtain the Weyl dimension formula:

$$\dim V_\lambda = \prod_{\alpha \in \Delta^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle}.$$

We can also think  $T$ -equivariantly for the natural torus action on  $X$ . In this case, the equation

$$\int_X p = \frac{\sum_{w \in W} (-1)^w w \cdot p}{\prod_{\alpha \in \Delta^+} \alpha},$$

gives the integral in equivariant cohomology, valued in  $H_T(\text{pt}) \cong \text{Sym}^\bullet(\mathfrak{t}_\mathbb{Q}^*)$ . We can then interpret (9) in the completion of this  $T$ -equivariant cohomology:

$$\int_X \text{ch}(\mathcal{L}_\lambda) \text{td}(T_X) = \frac{1}{\prod_{\alpha \in \Delta^+} \alpha} \sum_{w \in W} (-1)^w \frac{e^{w(\lambda + \rho)} \prod_{\alpha \in \Delta^+} \alpha}{\prod_{\alpha \in \Delta^+} e^{\alpha/2} - e^{-\alpha/2}} = \frac{\sum_{w \in W} (-1)^w e^{w(\lambda + \rho)}}{\prod_{\alpha \in \Delta^+} e^{\alpha/2} - e^{-\alpha/2}}.$$

The Chern character of  $V_\lambda$ , considered as a  $T$ -equivariant coherent sheaf on a point, is the sum over weight spaces  $\sum_\mu \dim(V_\lambda)_\mu e^\mu$ , so from equivariant GHRR, we obtain the usual Weyl character formula:

$$\sum_\mu \dim(V_\lambda)_\mu e^\mu = \frac{\sum_{w \in W} (-1)^w e^{w(\lambda + \rho)}}{\prod_{\alpha \in \Delta^+} e^{\alpha/2} - e^{-\alpha/2}}.$$

### 3. The function-sheaf correspondence

**3.1. Euler characteristic.** Instead of coherent sheaves, we can also consider constructible sheaves. Let  $k$  be a commutative ring, and for any topological space  $X$ , we let  $k_X$  denote the sheaf of locally constant<sup>3</sup>  $k$ -valued functions on  $U_i$ .

**DEFINITION 3.1.** We call a sheaf of  $k$ -modules  $\mathcal{F}$  on a topological space  $X$  a **local system** if it is locally constant, i.e. if there is a finite open cover  $\{U_i\}$ , such that  $\mathcal{F}|_{U_i} \cong k_{U_i}^{\oplus m}$  for some integer  $m$ .

A great example of a local system is the flat sections of a vector bundle  $E$  with a flat connection on a compact manifold. While globally there may be no sections, there is always an open cover  $U_i$  where for any  $u \in U_i$ , each element of the fiber  $E_u$  extends uniquely to a covariantly constant section of  $E$  over  $U_i$ . Thus a local system is a sheaf where nearby fibers are isomorphic, but in order to make this identification canonical, we have to shrink to a smaller neighborhood (and the identification may depend on the neighborhood we choose).

Now, assume that the topological space we consider is a complex quasi-projective variety. This space has two natural topologies both induced from the embedding in projective space: the **Zariski topology**, whose generating open sets are the locus where some meromorphic function on projective space has neither a zero nor a pole, and the **classical topology** which is induced by the usual smooth manifold structure on  $\mathbb{C}P^n$ . Constructible sheaves are sheaves of finitely generated  $k$ -modules on algebraic varieties that are locally modeled on local systems in the classical topology. We can define them inductively by saying that a sheaf  $\mathcal{F}$  on a variety of dimension  $n$  is **constructible** if there is a Zariski open subset  $U$  such that  $\mathcal{F}|_U$  is a local system in the classical topology and on the complementary Zariski closed subset  $V$  (of lower dimension),  $\mathcal{F}|_V$  is constructible<sup>4</sup>. We let  $\text{Sh}(X)$  be the category of constructible sheaves on  $X$ , and  $D^b\text{Sh}(X)$  the subcategory of the bounded derived category of all sheaves of vector spaces where all complexes have constructible cohomology.

Certain aspects of constructible life are actually much simpler than coherent sheaves. For example, instead of an index formula valued in homology we obtain one valued in the space of constructible functions on  $X$ ; as with sheaves, we inductively define **constructible functions** on a variety of dimension  $n$  to be those constant on a Zariski open subset  $U$ , with the restriction to the complement constructible. Let  $C(X)$  be the ring of constructible functions on  $X$  (with pointwise

<sup>3</sup>The presheaf that assigns the constant  $k$ -valued functions to any open subset is not a sheaf, since if a subset  $U = U_1 \cup U_2$  is the union of two open subsets with  $U_1 \cap U_2 = \emptyset$ , then, a function that takes value  $a$  on  $U_1$  and  $b$  on  $U_2$  must define a section of the sheafification by the gluing property. The locally constant functions are, essentially by definition, the sheafification of this presheaf.

<sup>4</sup>Note the odd mix of the classical and Zariski topologies here; this will be eased a bit when we consider the étale topology.