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LÁSZLÓ ERDŐS
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A Dynamical Approach to Random Matrix Theory

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A Dynamical Approach to Random Matrix Theory

LÁSZLÓ ERDŐS AND HORNG-TZER YAU

This book is a concise and self-contained introduction of recent techniques to prove local spectral universality for large random matrices. Random matrix theory is a fast expanding research area, and this book mainly focuses on the methods that the authors participated in developing over the past few years. Many other interesting topics are not included, and neither are several new developments within the framework of these methods. The authors have chosen instead to present key concepts that they believe are the core of these methods and should be relevant for future applications. They keep technicalities to a minimum to make the book accessible to graduate students. With this in mind, they include in this book the basic notions and tools for high-dimensional analysis, such as large deviation, entropy, Dirichlet form, and the logarithmic Sobolev inequality.

This manuscript has been developed and continuously improved over the last five years. The authors have taught this material in several regular graduate courses at Harvard, Munich, and Vienna, in addition to various summer schools and short courses.

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A Dynamical Approach to Random Matrix Theory

Courant Lecture Notes in Mathematics

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Preface

This book is a concise and self-contained introduction to the recent techniques to prove local spectral universality for large random matrices. Random matrix theory is a fast-expanding research area, and this book mainly focuses on the methods we participated in developing over the past few years. Many other interesting topics are not included, nor are several new developments within the framework of these methods. We have chosen instead to present key concepts that we believe are the core of these methods and should be relevant for future applications. We keep technicalities to a minimum to make the book accessible to graduate students. With this in mind, we include in this book the basic notions and tools for high-dimensional analysis such as large deviation, entropy, Dirichlet form, and logarithmic Sobolev inequality.

The material in this book originates from our joint works with a group of collaborators over the past several years. Not only were the main mathematical results in this book taken from these works, but the presentation of many sections followed the routes laid out in these papers. In alphabetical order, these coauthors were Paul Bourgade, Antti Knowles, Sandrine Péché, Jose Ramírez, Benjamin Schlein, and Jun Yin. We would like to thank all of them.

This manuscript was developed and continuously improved over the last five years. We have taught this material in several regular graduate courses at Harvard, Munich, and Vienna, in addition to various summer schools and short courses. We are thankful for the generous support of the Institute for Advanced Study, Princeton, where part of this manuscript was written during the special year devoted to random matrices in 2013–14. L.E. also thanks Harvard University for the continuous support during his numerous visits. L.E. was partially supported by the SFB TR 12 grant of the German Science Foundation and the ERC Advanced Grant, RANMAT 338804 of the European Research Council. H.-T. Y. would like to thank the National Center for the Theoretic Sciences at the National Taiwan University, where part of the manuscript was written, for the hospitality and support for his several visits. H.-T. Y. gratefully acknowledges the support from NSF DMS-1307444 and DMS-1606305 and a Simons Investigator award.

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CHAPTER 1

Introduction

Perhaps I am now too courageous when I try to guess the distribution of the distances between successive levels [of energies of heavy nuclei].... Theoretically, the situation is quite simple if one attacks the problem in a simple-minded fashion. The question is simply 'what are the distances of the characteristic values of a symmetric matrix with random coefficients?'

Eugene Wigner on the Wigner surmise, 1956 [**71**]

Random matrices appeared in the literature as early as 1928, when Wishart [**139**] used them in statistics. The natural question regarding their eigenvalue statistics, however, was not raised until the pioneering work [**138**] of Eugene Wigner in the 1950s. Wigner's original motivation came from nuclear physics when he noticed from experimental data that gaps in energy levels of large nuclei tend to follow the same statistics irrespective of the material. Quantum mechanics predicts that energy levels are eigenvalues of a self-adjoint operator, but the correct Hamiltonian operator describing nuclear forces was not known at that time. In addition, the computation of the energy levels of large quantum systems would have been impossible even with the full Hamiltonian explicitly given. Instead of pursuing a direct solution to this problem, Wigner appealed to a phenomenological model to explain his observation. Wigner's pioneering idea was to model the complex Hamiltonian by a random matrix with independent entries. All physical details of the system were ignored except one, the *symmetry type*: systems with time-reversal symmetry were modeled by real symmetric random matrices, while complex Hermitian random matrices were used for systems without time-reversal symmetry (e.g., with magnetic forces). This simple-minded model amazingly reproduced the correct gap statistics, indicating a profound universality principle working in the background.

Notwithstanding their physical significance, random matrices are also very natural mathematical objects, and their studies could have been initiated by mathematicians driven by pure curiosity. A large number of random numbers and vectors have been known to exhibit universal patterns; the obvious examples are the law of large numbers and the central limit theorem. What are their analogues in the noncommutative setting, e.g., for matrices? Focusing on the spectrum, what do eigenvalues of typical large random matrices look like?

As the first result of this type, Wigner proved a type of law of large numbers for the density of eigenvalues, which we now explain. The (real or complex) Wigner ensembles consist of $N \times N$ self-adjoint matrices $H = (h_{ij})$ with matrix elements having mean zero and variance $1/N$ that are independent up to the symmetry constraint $h_{ij} = \overline{h}_{ji}$. The Wigner semicircle law states that the empirical density of the eigenvalues of H is given by the semicircle law, $\varrho_{\mathrm{sc}}(x) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}$, as $N \to \infty$, independent of the details of the distribution of h_{ij}.

On the scale of individual eigenvalues, Wigner predicted that the fluctuations of the gaps are universal and their distribution is given by a new law, the *Wigner surmise*. This might be viewed as the random matrix analogue of the central limit theorem.

After Wigner's discovery, Dyson, Gaudin, and Mehta achieved several fundamental mathematical results. In particular, they were able to compute the gap distribution and the local correlation functions of the eigenvalues for Gaussian ensembles. They are called the *Gaussian orthogonal ensemble* (GOE) and the *Gaussian unitary ensemble* (GUE), corresponding to the two most important symmetry classes, the real symmetric and complex Hermitian matrices. The Wigner surmise turned out to be slightly wrong and the correct law is given by the *Gaudin distribution*. Dyson and Mehta gradually formulated what is nowadays known as the *Wigner-Dyson-Mehta* (WDM) universality conjecture. As presented in the classical treatise of Mehta [**106**], this conjecture asserts that the local eigenvalue statistics for large random matrices with independent entries are universal; i.e., they do not depend on the particular distribution of the matrix elements. In particular, they coincide with those in the Gaussian case that were computed explicitly.

On a more philosophical level, we can recast Wigner's vision as the hypothesis that the eigenvalue gap distributions for large complicated quantum systems are universal in the sense that they depend only on the symmetry class of the physical system but not on other detailed structures. Therefore, the Wigner-Dyson-Mehta universality conjecture is merely a test of Wigner's hypothesis for a special class of matrix models, the *Wigner ensembles*, which are characterized by the independence of their matrix elements. The other large class of matrix models is the *invariant ensembles*. They are defined by a Radon-Nikodym density of the form $\exp(-\mathrm{Tr}\, V(H))$ with respect to the flat Lebesgue measure on the space of real symmetric or complex Hermitian matrices H. Here V is a real-valued function called the *potential* of the invariant ensemble. These distributions are invariant under orthogonal or unitary conjugation, but the matrix elements are not independent except for the Gaussian case. The universality conjecture for invariant ensembles asserts that the local eigenvalue gap statistics are independent of the function V.

In contrast to the Wigner case, substantial progress was made for invariant ensembles in the last two decades. The key element of this success was that invariant ensembles, unlike Wigner matrices, have explicit formulas for

the joint densities of the eigenvalues. These formulas express the eigenvalue correlation functions as determinants whose entries are given by functions of orthogonal polynomials. The limiting local eigenvalue statistics are thus determined by the asymptotic behavior of these formulas and in particular those of the orthogonal polynomials as the size of the matrix tends to infinity. A key important tool, the Riemann-Hilbert method [**72**], was brought into this subject by Fokas, Its, and Kitaev, and the universality of eigenvalue statistics was established for large classes of invariant ensembles by Bleher-Its [**20**] and by Deift and collaborators [**37–41**].

Behind the spectacular progress in understanding the local statistics of invariant ensembles, the cornerstone of this approach lies in the fact that there are explicit formulas to represent eigenvalue correlation functions by orthogonal polynomials—a key observation made in the original work of Gaudin and Mehta for Gaussian random matrices. For Wigner ensembles, there are no explicit formulas for the eigenvalue statistics, and the WDM conjecture was open for almost 50 years with virtually no progress. The first significant advance in this direction was made by Johansson [**86**], who proved the universality for *complex* Hermitian matrices under the assumption that the common distribution of the matrix entries has a substantial Gaussian component; i.e., the random matrix H is of the form $H = H_0 + aH^{\mathrm{G}}$ where H_0 is a general Wigner matrix, H^{G} is the GUE matrix, and a is a certain, not too small, positive constant independent of N. His proof relied on an explicit formula by Brézin and Hikami [**30, 31**] that uses a certain version of the Harish-Chandra-Itzykson-Zuber formula [**85**]. These formulas are available for the complex Hermitian case only, which restricted the method to this symmetry class.

If local spectral universality is so ubiquitous as Wigner, Dyson, and Mehta conjectured, there must be an underlying mechanism driving local statistics to their universal fixed point. In hindsight, the existence of such a mechanism is almost a synonym of "universality." However, up to ten years ago there was no clear indication whether a solution of the WDM conjecture would rely on some yet-to-be discovered formulas or would come from some other deep insight.

The goal of this book is to give a self-contained introduction to a new approach to local spectral universality which, in particular, resolves the WDM conjecture. To keep technicalities to a minimum, we will consider the simplest class of matrices that we call *generalized Wigner matrices* (Definition 2.1), and we will focus on proving universality in a simpler form, in the *averaged energy* sense (see Section 5.1) and the necessary prerequisites. We stress that these are not the strongest results that are currently available, but they are good representatives of the general ideas we have developed in the past several years. We believe that these techniques are sufficiently mature to be presented in a book format.

This approach consists of the following three steps:

Step 1. *Local semicircle law*: It provides an a priori estimate showing that the density of eigenvalues of generalized Wigner matrices is given by the semicircle law at very small microscopic scales, i.e., down to spectral intervals that contain N^{ε} eigenvalues.

Step 2. *Universality for Gaussian divisible ensembles*: It proves that the local statistics of *Gaussian divisible ensembles* $H_0 + aH^{\mathrm{G}}$ are the same as those of the Gaussian ensembles H^{G} as long as $a \geq N^{-1/2+\varepsilon}$, i.e., already for very small a.

Step 3. *Approximation by a Gaussian divisible ensemble*: It is a type of "density argument" that extends the local spectral universality from Gaussian divisible ensembles to all Wigner ensembles.

The conceptually novel point of our method is Step 2. The eigenvalue distributions of the Gaussian divisible ensembles, written in the form $e^{-t/2}H_0 + \sqrt{1-e^{-t}}H^{\mathrm{G}}$, are the same as that of the solution of a *matrix-valued Ornstein-Uhlenbeck* (OU) process H_t for any time $t \geq 0$. Dyson [**45**] observed half a century ago that the dynamics of the eigenvalues of H_t is given by an interacting stochastic particle system, called the *Dyson Brownian motion* (DBM). In addition, the invariant measure of this dynamics is exactly the eigenvalue distribution of GOE or GUE. This invariant measure is also a Gibbs measure of point particles in one dimension interacting via a long-range, logarithmic potential. Using a heuristic physical argument, Dyson remarked [**45**] that the DBM reaches its "local equilibrium" on a short time scale $t \gtrsim N^{-1}$. We will refer to this as *Dyson's conjecture*, although it was more an intuitive physical picture than an exact mathematical statement.

Since Dyson's work in the 1960s, there has been virtually no progress in proving this conjecture. Besides the limit of available mathematical tools, one main factor is the vague formulation of the conjecture involving the notion of "local equilibrium," which even nowadays is not well-defined for a Gibbs measure with a general long-range interaction. Furthermore, a possible connection between Dyson's conjecture and the solution of the WDM conjecture has never been elucidated in the literature.

In fact, "relaxation to local equilibrium" in a time scale t refers to the phenomenon that after time t the dynamics has changed the system, at least locally, from its initial state to a local equilibrium. It therefore appears counterintuitive that one may learn anything useful in this way about the WDM conjecture, since the latter concerns the initial state. The key point is that by applying local relaxation to *all* initial states (within a reasonable class) simultaneously, Step 2 generates a large set of random matrix ensembles for which universality holds. We prove that, for the purpose of universality, this set is sufficiently dense so that any Wigner matrix H is sufficiently close to a Gaussian divisible ensemble of the form $e^{-t/2}H_0 + \sqrt{1-e^{-t}}H^{\mathrm{G}}$ with a suitably chosen H_0. Originally, our motivation was driven not by Dyson's conjecture, but by the desire to prove the

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