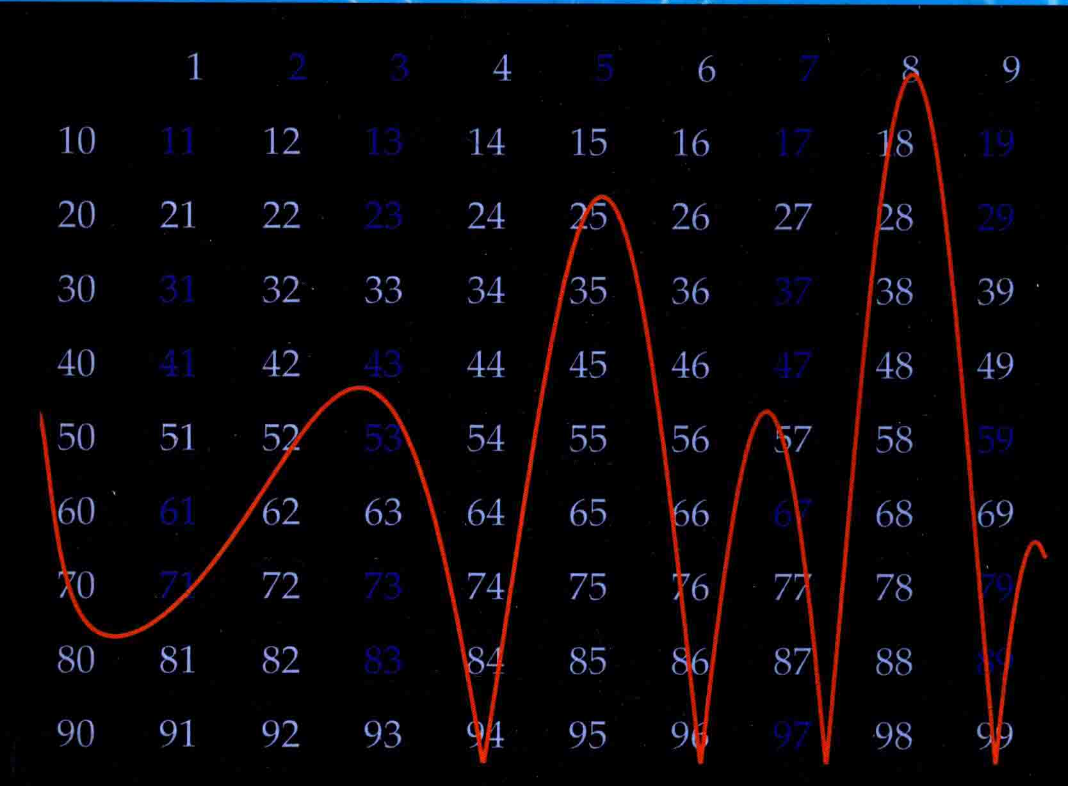
THE RIEMANN HYPOTHESIS



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The Riemann Hypothesis

A Million Dollar Problem

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MAA

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The half title page figure shows the graph of the function

$$f(y) = |\zeta(\frac{1}{2} + i\,y)|$$

for $0 \leq y \leq 32$ against a background in which the primes less than 100 are marked (see also figure 3.7 on page 54).

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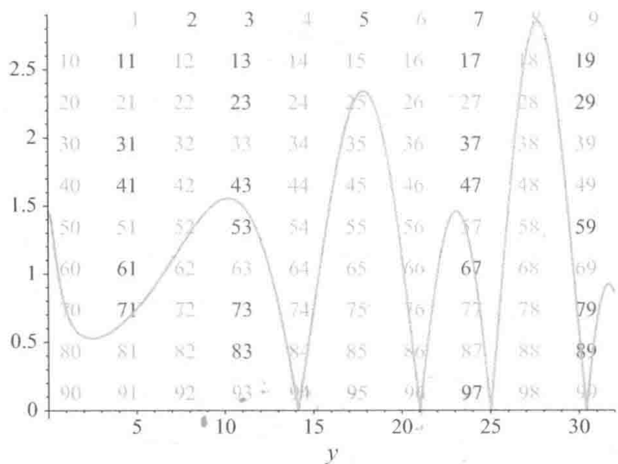

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Preface

Mathematics is full of unsolved problems and other mysteries, but none more important and intriguing than the *Riemann hypothesis*. Baffling the greatest minds for more than a hundred and fifty years, the Riemann hypothesis is at the very core of mathematics. A proof of it would mean an enormous advance. In addition, the Riemann hypothesis was chosen as one of the seven *Millennium Problems*[1] by the Clay Mathematics Institute. This means that proving the Riemann hypothesis will not only make you world famous, but also earns you a one million dollar prize.

The Riemann hypothesis concerns the *prime numbers*

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \ldots$$

i.e., the integer numbers greater than 1 that are not divisible by any smaller number (except 1). Ubiquitous and fundamental in mathematics as they are, it is important and interesting to know as much as possible about these numbers. Simple questions would be: how are the prime numbers distributed among the positive integers? How many prime numbers are there? What is the number of prime numbers of one hundred digits? Of one thousand digits? These questions were the starting point of a groundbreaking paper by Bernhard Riemann written in 1859. As an aside in his article Riemann formulated his now famous hypothesis, that so far nobody has come close to proving:

The Riemann Hypothesis. *All nontrivial zeroes of the zeta function lie on the critical line.*

Hidden behind this at first mysterious statement, lies a whole mathematical universe of prime numbers, infinite sequences, infinite products and complex functions. The present book is a first exploration of this fascinating world.

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[1] see http://www.claymath.org/millennium/Riemann_Hypothesis/

The Riemann hypothesis as an online course

The Riemann hypothesis was the subject of a *web class*, a four-week online course organized four times in the years 2006 to 2010 by the Korteweg-De Vries Institute for Mathematics at the University of Amsterdam. It was aimed at mathematically talented secondary school students. This book originated from the course material. The goal of these courses was not to solve the Riemann hypothesis, but to introduce interested and talented students to challenging university level mathematics. At the end of the course the participants had a good idea of what the Riemann hypothesis is and why it is such an important and interesting problem in mathematics. After attending the web class, many of them decided upon university study in mathematics.

A web class at the University of Amsterdam is a four-week course in which secondary school students from all over the country work at home and communicate with each other and with university staff via the internet. They study the course material, solve the exercises, send their solutions to the staff and get feedback within a few days.

The four chapters of this book correspond to the material of the four weeks of the web class, one for every week. For some of the exercises the participants could use computer algebra worksheets made available by the university. The readers of this book may use free mathematical software on the internet instead. In the appendix *Computer support* on page 91 we give suggestions and instructions.

During the web class there was much interest in the material by mathematics teachers, secondary school students and undergraduate university students. Therefore we decided to make it available as a book, including full solutions to all exercises, and extended with quite a few challenging additional exercises at the end of each chapter. The latter are somewhat more difficult and may be skipped on first reading. To maximize accessibility, we have limited the prerequisites for the text to a minimal amount of elementary calculus, excluding integration.

How to read a mathematics text?

Mathematics texts are usually written in a very concise manner: each and every detail counts. This means that they can never be read both fast and thoroughly at the same time. Fast reading is possible, but only to get a first impression. Thereafter you should reread, focussing on more details. And then again and again, if necessary. If you get stuck, our advice is simple: read on and come back later. The exercises are the ultimate test: if you can solve them, you understand the matter. Do not look at the solution before trying to solve an exercise on your own, but afterwards always compare your solution with the one given at the end of the book.

Contents

1

Prime numbers

How are the prime numbers distributed among the other numbers? How many primes are there? What is the number of primes of one hundred digits? Of one thousand digits? These questions were the starting point of a seminal paper by Bernhard Riemann (1826–1866) written in 1859. As an aside in his article Riemann formulated his now famous hypothesis that so far nobody has come close to proving.

In this first chapter you will get to know the primes, two distinct functions that count primes, and various approximations of these functions. The natural logarithm plays an important role. At the end of the chapter the Riemann hypothesis itself will make its first appearance.

1.1 Primes as elementary building blocks

Counting is more than twenty thousand years old. Long before written language was invented, people already were tallying with notches on pieces of bone[1]. Counting and arithmetic are arguably the oldest concepts in mathematics and yet numbers still exhibit mysterious patterns that we do not fully understand. For example, if you ask how numbers can be constructed by multiplication alone, you quickly hit upon one of the biggest riddles of mathematics.

Using a single number as a building block, you can never construct all numbers by multiplication only. Take 2 for example. All you can get is 2 itself, $2 \times 2 = 4$, $2 \times 2 \times 2 = 8$, 16, 32, ... and the other powers of 2. If you add 3 as a building block, then you can also construct 3, $2 \times 3 = 6$, 9, 12, ... However, the number 5 is still out of reach. By adding 5 to the list it is possible to construct all marked numbers in table 1.1.

[1] The so-called Ishango bone, see, e.g., nrich.maths.org/6013

	1	**2**	**3**	**4**	**5**	**6**	7	**8**	**9**
10	11	**12**	13	14	**15**	**16**	17	**18**	19
20	21	22	23	**24**	**25**	26	**27**	28	29
30	31	**32**	33	34	35	**36**	37	38	39
40	41	42	43	44	**45**	46	47	**48**	49
50	51	52	53	**54**	55	56	57	58	59
60	61	62	63	**64**	65	66	67	68	69
70	71	**72**	73	74	**75**	76	77	78	79
80	**81**	82	83	84	85	86	87	88	89
90	91	92	93	94	95	**96**	97	98	99

Table 1.1. The numbers 2, 3 and 5 as building blocks for multiplication.

Adding 7, the first number not yet constructed, as a building block, more numbers come within reach, but 11 and 13 still elude us. Will this process finally come to an end, or are we forced to add ever more building blocks to construct all numbers by multiplication?

The ancient Greeks already knew the answer to this question: there is no end to the list of building blocks. The reason is that at any given moment there are special numbers that clearly cannot be constructed with the building blocks you have at that time. For example, suppose your building blocks are just 2, 3, 5 and 7. The special number that will certainly be out of reach with these blocks is $211 = 2 \times 3 \times 5 \times 7 + 1$. Why? Well, first of all it is not divisible by any of the building blocks since division by 2, 3, 5 or 7 leaves a remainder of 1. However, all constructible numbers must be divisible by at least one of the four building blocks.

The numbers 2, 3, 5, 7, 11, ... are called the *prime numbers* (or *primes*, for short). They are the elementary building blocks from which all numbers can be constructed by multiplication. A prime cannot be divisible by any smaller number except 1, since otherwise it would not be necessary as a building block. The argument in the previous paragraph shows that the list of prime numbers never ends. In other words: *there are infinitely many prime numbers*. The number 1 is a bit different. It is of little use as a building block since $1 \times n = n$ for all numbers n. This, among other things, causes mathematicians not to include 1 in the list of primes.

Exercise 1.1. a. Using Table 1.1 find all primes smaller than 100.

b. What numbers smaller than 100 can you construct by multiplication, using only the numbers 3 and 8 as building blocks?

c. Is it possible to construct $103 = 2 \times 3 \times 17 + 1$ using only 2, 3 and 17? What about 104?

d. Is the number $2 \times 3 \times 5 \times 7 \times 11 + 1$ prime? And what about the number $2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1$?

1.2 Counting primes

How are the primes distributed among the other numbers? That is what this book is all about. We already know that there are infinitely many primes, but that does not tell much about their distribution. For some other infinite sequences of numbers it is easy to see how they are distributed. The odd numbers for example. Although there are infinitely many of them, we know exactly how they are distributed among the other numbers: every other number is odd.

With the primes things are more subtle. The sequence of prime numbers is not caught as easily in a regular pattern or in a formula. It is easy to say what the thousandth odd number is (as an aside: what is the thousandth odd number?), but it is another matter if you want to know the thousandth prime. All you can do is run through the sequence of primes one by one. Nobody can predict what the next prime will be. And yet we will see that there exists a mysterious sense of order in the seemingly erratic sequence of primes.

This combination of unpredictability and regularity is especially striking when you count the number of primes below a certain bound. From exercise 1.1 we know that there are exactly 25 primes below 100, but how many are there below 200? And how many primes are less than 1000? To study this question systematically we denote the number of primes less than or equal to x by $\pi(x)$. For example, $\pi(100) = 25$ because there are exactly 25 primes less than or equal to 100; $\pi(2) = 1$ since there is only a single prime less than or equal to 2, the number 2 itself. $\pi(10) = 4$, because there are four prime numbers below 10; these are 2, 3, 5 and 7. Finally $\pi(17.351) = 7$, since there are seven prime numbers less than or equal to 17.351, namely 2, 3, 5, 7, 11, 13 and 17. The function $\pi(x)$ is called the *prime counting function* because it counts the number of primes in a certain sense. It is a function that is defined on all real numbers (not just on the natural numbers). The graph of $\pi(x)$ is a sort of staircase that makes a jump of 1 above every prime number.

In this chapter we will also meet other counting functions. Be careful: the notation $\pi(x)$ has nothing to do with the famous number $\pi = 3.1415926\ldots$. In mathematics the same symbol sometimes has several different meanings. The intended meaning is usually clear from the context.

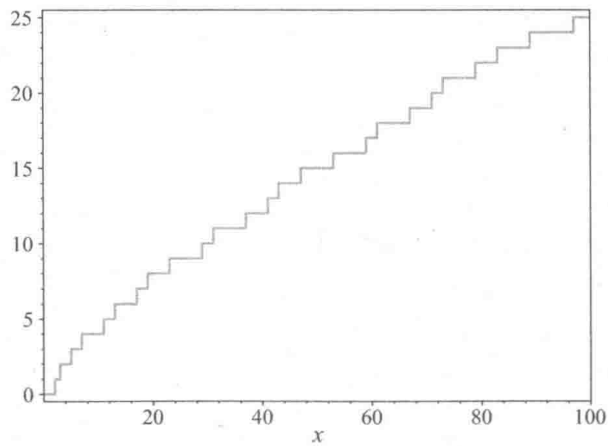

Figure 1.1. The prime counting function $\pi(x)$.

Although there is no easy formula for the function $\pi(x)$, it is still possible to graph it and gain valuable information on the prime numbers. Figure 1.1 shows a graph of $\pi(x)$ for $0 \leq x \leq 100$. Using the computer it is easy to further examine the graph of $\pi(x)$. This is the subject of the next exercise. For suggestions on how to use the computer to complete this and future exercises, see the appendix *Computer support* on page 91.

Exercise 1.2. a. Using the computer, examine graphs of $\pi(x)$ on various domains. First take $0 \leq x \leq 100$, as in Figure 1.1. This graph should also be consistent with the outcome of exercise 1.1.a. Check this!

b. Also try the domains $0 \leq x \leq 1\,000$, $0 \leq x \leq 10\,000, \ldots,$ $0 \leq x \leq 1\,000\,000$.

c. Find the smallest prime number greater than 1 000 000.

d. What is the ten-thousandth prime number?

From nearby, the graph of $\pi(x)$ may look like a chaotic staircase, but by choosing larger and larger domains you will have noticed a beautiful regularity. The smooth, almost straight graph contains many of the secrets of the primes. To unlock these secrets we will try to find formulas to describe the graph. Is there a simple function whose graph looks like that of $\pi(x)$, at least when viewed from a distance?

Here are some experiments using the computer to find approximations of $\pi(x)$. The approximating functions we tried are linear functions of the

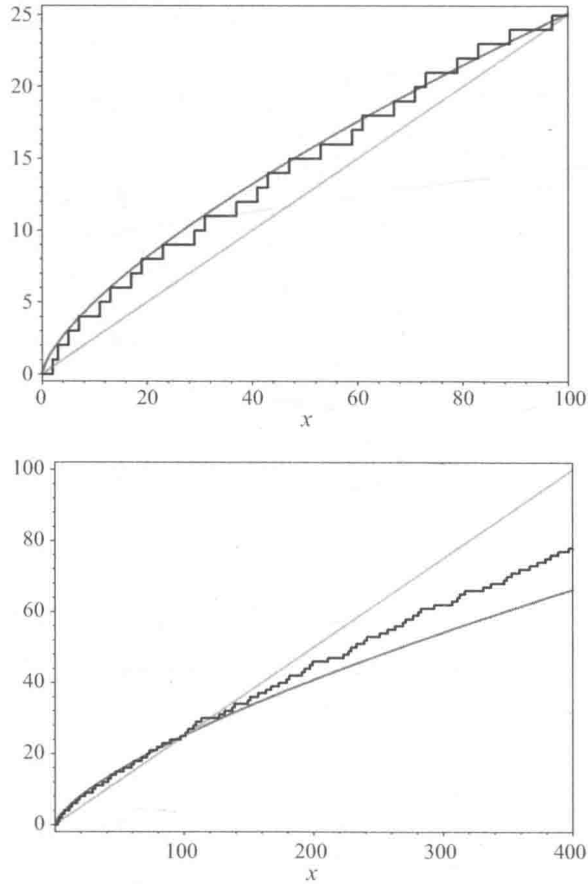

Figure 1.2. The prime counting function $\pi(x)$ and the approximations $0.25\,x$ and $x^{0.7}$ on the interval [0, 100] (top) and on the interval [0, 400] (bottom).

form cx and power functions of the form x^a. In Figure 1.2 you see the graph of $\pi(x)$ together with the approximations $0.25\,x$ and $x^{0.7}$. On the interval [0, 100] both approximations look pretty good but on a larger interval they fail completely. The first graph increases way too fast and the second approximation does not grow quickly enough.

Figure 1.3 shows our second attempt at approximation. On the large interval [0, 10000] the functions $0.125\,x$ and $x^{0.773}$ seem good approximations but again on the larger interval [0, 40000] they fail miserably.

What is striking in Figure 1.3 is that $\pi(x)$ now seems just as smooth as the other two graphs. The graph of $\pi(x)$ increases but the growth is slower and slower. This is not strange, since the bigger a number the more possible