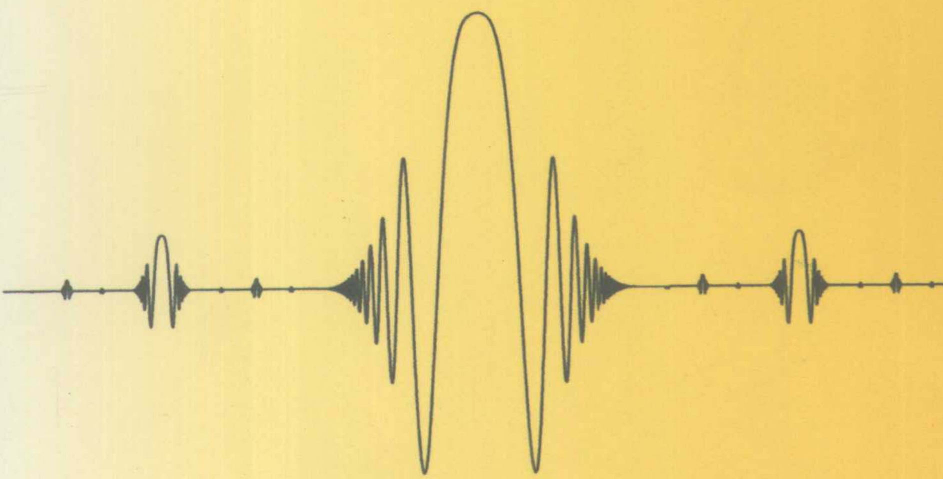
Undergraduate Texts in Mathematics

Stephen Abbott

UNDERSTANDING ANALYSIS

分析入门



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Preface

My primary goal in writing *Understanding Analysis* was to create an elementary one-semester book that exposes students to the rich rewards inherent in taking a mathematically rigorous approach to the study of functions of a real variable. The aim of a course in real analysis should be to challenge and improve mathematical intuition rather than to verify it. There is a tendency, however, to center an introductory course too closely around the familiar theorems of the standard calculus sequence. Producing a rigorous argument that polynomials are continuous is good evidence for a well-chosen definition of continuity, but it is not the reason the subject was created and certainly not the reason it should be required study. By shifting the focus to topics where an untrained intuition is severely disadvantaged (e.g., rearrangements of infinite series, nowhere-differentiable continuous functions, Fourier series), my intent is to restore an intellectual liveliness to this course by offering the beginning student access to some truly significant achievements of the subject.

The Main Objectives

In recent years, the standard undergraduate curriculum in mathematics has been subjected to steady pressure from several different sources. As computers and technology become more ubiquitous, so do the areas where mathematical thinking can be a valuable asset. Rather than preparing themselves for graduate study in pure mathematics, the present majority of mathematics majors look forward to careers in banking, medicine, law, and numerous other fields where analytical skills are desirable. Another strong influence on college mathematics is the ongoing calculus reform effort, now well over ten years old. At the core of this movement is the justifiable goal of presenting calculus in a more intuitive way, emphasizing geometric arguments over symbolic ones. Despite these various trends—or perhaps because of them—nearly every undergraduate mathematics program continues to require at least one semester of real analysis. The result is that instructors today are faced with the task of teaching a difficult, abstract course to a more diverse audience less familiar with the nature of axiomatic arguments.

The crux of the matter is that any prevailing sentiment in favor of marketing mathematics to larger groups must at some point be reconciled with the fact that theoretical analysis is extremely challenging and even intimidating for some. One unfortunate resolution of this dilemma has been to make the course easier by making it less interesting. The omitted material is inevitably what gives analysis its true flavor. A better solution is to find a way to make the more advanced topics accessible and worth the effort.

I see three essential goals that a semester of real analysis should try to meet:

1. Students, especially those emerging from a reform approach to calculus, need to be convinced of the need for a more rigorous study of functions. The necessity of precise definitions and an axiomatic approach must be carefully motivated.

2. Having seen mainly graphical, numerical, or intuitive arguments, students need to learn what constitutes a rigorous mathematical proof and how to write one.

3. There needs to be significant reward for the difficult work of firming up the logical structure of limits. Specifically, real analysis should not be just an elaborate reworking of standard introductory calculus. Students should be exposed to the tantalizing complexities of the real line, to the subtleties of different flavors of convergence, and to the intellectual delights hidden in the paradoxes of the infinite.

The philosophy of *Understanding Analysis* is to focus attention on questions that give analysis its inherent fascination. Does the Cantor set contain any irrational numbers? Can the set of points where a function is discontinuous be arbitrary? Are derivatives continuous? Are derivatives integrable? Is an infinitely differentiable function necessarily the limit of its Taylor series? In giving these topics center stage, the hard work of a rigorous study is justified by the fact that *they are inaccessible without it.*

The Structure of the Book

This book is an introductory text. Although some fairly sophisticated topics are brought in early to advertise and motivate the upcoming material, the main body of each chapter consists of a lean and focused treatment of the core topics that make up the center of most coursès in analysis. Fundamental results about completeness, compactness, sequential and functional limits, continuity, uniform convergence, differentiation, and integration are all incorporated. What is specific here is where the emphasis is placed. In the chapter on integration, for instance, the exposition revolves around deciphering the relationship between continuity and the Riemann integral. Enough properties of the integral are obtained to justify a proof of the Fundamental Theorem of Calculus, but the theme of the chapter is the pursuit of a characterization of integrable functions in terms of continuity. Whether or not Lebesgue's measure-zero criterion is treated, framing the material in this way is still valuable because it is the questions that are important. Mathematics is not a static discipline. Students should be aware of the historical reasons for the creation of the mathematics they are learning and by extension realize that there is no last word on the subject. In the case of integration, this point is made explicitly by including some relatively recent developments on the generalized Riemann integral in the additional topics of the last chapter.

The structure of the chapters has the following distinctive features.

Discussion Sections: Each chapter begins with the discussion of some motivating examples and open questions. The tone in these discussions is intentionally informal, and full use is made of familiar functions and results from calculus. The idea is to freely explore the terrain, providing context for the upcoming definitions and theorems. A recurring theme is the resolution of the paradoxes that arise when operations that work well in finite settings are naively extended to infinite settings (e.g., differentiating an infinite series term-by-term, reversing the order of a double summation). After these exploratory introductions, the tone of the writing changes, and the treatment becomes rigorously tight but still not overly formal. With the questions in place, the need for the ensuing development of the material is well-motivated and the payoff is in sight.

Project Sections: The penultimate section of each chapter (the final section is a short epilogue) is written with the exercises incorporated into the exposition. Proofs are outlined but not completed, and additional exercises are included to elucidate the material being discussed. The point of this is to provide some flexibility. The sections are written as self-guided tutorials, but they can also be the subject of lectures. I have used them in place of a final examination, and they work especially well as collaborative assignments that can culminate in a class presentation. The body of each chapter contains the necessary tools, so there is some satisfaction in letting the students use their newly acquired skills to ferret out for themselves answers to questions that have been driving the exposition.

Building a Course

Teaching a satisfying class inevitably involves a race against time. Although this book is designed for a 12–14 week semester, there are still a few choices to make as to what to cover.

- The introductions can be discussed, assigned as reading, omitted, or substituted with something preferable. There are no theorems proved here that show up later in the text. I do develop some important examples in these introductions (the Cantor set, Dirichlet's nowhere-continuous function) that probably need to find their way into discussions at some point.

- Chapter 3, Basic Topology of **R**, is much longer than it needs to be. All that is required by the ensuing chapters are fundamental results about open and closed sets and a thorough understanding of sequential compactness. The characterization of compactness using open covers as well as the section on perfect and connected sets are included for their own intrinsic interest. They are not, however, crucial to any future proofs. The one exception to this is a presentation of the Intermediate Value Theorem (IVT) as a special case of the preservation of connected sets by continuous functions. To keep connectedness truly optional, I have included two direct proofs of IVT, one using least upper bounds and the other using nested intervals. A similar comment can be made about perfect sets. Although proofs of the Baire Category Theorem are nicely motivated by the argument that perfect sets are uncountable, it is certainly possible to do one without the other.

- All the project sections (1.5, 2.8, 3.5, 4.6, 5.4, 6.6, 7.6, 8.1–8.4) are optional in the sense that no results in later chapters depend on material in these sections. The four topics covered in Chapter 8 are also written in this project-style format, where the exercises make up a significant part of the development. The only one of these sections that might require a lecture is the unit on Fourier series, which is a bit longer than the others.

The Audience

The only prerequisite for this course is a robust understanding of the results from single-variable calculus. The theorems of linear algebra are not needed, but the exposure to abstract arguments and proof writing that usually comes with this course would be a valuable asset. Complex numbers are never used in this book.

The proofs in *Understanding Analysis* are written with the introductory student firmly in mind. Brevity and other stylistic concerns are postponed in favor of including a significant level of detail. Most proofs come with a fair amount of discussion about the context of the argument. What should the proof entail? Which definitions are relevant? What is the overall strategy? Is one particular proof similar to something already done? Whenever there is a choice, efficiency is traded for an opportunity to reinforce some previously learned technique. Especially familiar or predictable arguments are usually sketched as exercises so that students can participate directly in the development of the core material.

The search for recurring ideas exists at the proof-writing level and also on the larger expository level. I have tried to give the course a narrative tone by picking up on the unifying themes of approximation and the transition from the finite to the infinite. To paraphrase a passage from the end of the book, real numbers are approximated by rational ones; values of continuous functions are approximated by values nearby; curves are approximated by straight lines; areas are approximated by sums of rectangles; continuous functions are approximated by polynomials. In each case, the approximating objects are tangible and well-understood, and the issue is when and how well these qualities survive the limiting process. By focusing on this recurring pattern, each successive topic builds on the intuition of the previous one. The questions seem more natural, and a method to the madness emerges from what might otherwise appear as a long list of theorems and proofs.

This book always emphasizes core ideas over generality, and it makes no effort to be a complete, deductive catalog of results. It is designed to capture the intellectual imagination. Those who become interested are then exceptionally well prepared for a second course starting from complex-valued functions on more general spaces, while those content with a single semester come away with a strong sense of the essence and purpose of real analysis. Turning once more to the concluding passages of Chapter 8, "By viewing the different infinities of mathematics through pathways crafted out of finite objects, Weierstrass and the other founders of analysis created a paradigm for how to extend the scope of mathematical exploration deep into territory previously unattainable."

This exploration has constituted the major thrill of my intellectual life. I am extremely pleased to offer this guide to what I feel are some of the most impressive highlights of the journey. Have a wonderful trip!

Acknowledgments

The genesis of this book came from an extended series of conversations with Benjamin Lotto of Vassar College. The structure of the early chapters and the book's overall thesis are in large part the result of several years of sharing classroom notes, ideas, and experiences with Ben. I am pleased with how the manuscript has turned out, and I have no doubt that it is an immeasurably better book because of Ben's early contributions.

A large part of the writing was done while I was enjoying a visiting position at the University of Virginia. Special thanks go to Nat Martin and Larry Thomas for being so generous with their time and wisdom, and especially to Loren Pitt, the scope of whose advice extends well beyond the covers of this book. I would also like to thank Julie Riddleberger for her help with many of the figures. Marian Robbins of Bellarmine College, Steve Kennedy of Carleton College, Paul Humke of Saint Olaf College, and Tom Kriete of the University of Virginia each taught from a preliminary draft of this text. I appreciate the many suggested improvements that this group provided, and I want to especially acknowledge Paul Humke for his contributions to the chapter on integration.

My department and the administration of Middlebury College have also been very supportive of this endeavor. David Guertin came to my technological rescue on numerous occasions, Priscilla Bremser read early chapter drafts, and Rick Chartrand's insightful opinions greatly improved some of the later sections. The list of students who have suffered through the long evolution of this book is now too long to present, but I would like to mention Brooke Sargent, whose meticulous class notes were the basis of the first draft, and Jesse Johnson, who has worked tirelessly to improve the presentation of the many exercises in the book. The production team at Springer has been absolutely first-rate. My sincere thanks goes to all of them with a special nod to Sheldon

Axler for encouragement and advice surely exceeding anything in his usual job description.

In a recent rereading of the completed text, I was struck by how frequently I resort to historical context to motivate an idea. This was not a conscious goal I set for myself. Instead, I feel it is a reflection of a very encouraging trend in mathematical pedagogy to humanize our subject with its history. From my own experience, a good deal of the credit for this movement in analysis should go to two books: *A Radical Approach to Real Analysis*, by David Bressoud, and *Analysis by Its History*, by E. Hairer and G. Wanner. Bressoud's book was particularly influential to the presentation of Fourier series in the last chapter. Either of these would make an excellent supplementary resource for this course. While I do my best to cite their historical origins when it seems illuminating or especially important, the present form of many of the theorems presented here belongs to the common folklore of the subject, and I have not attempted careful attribution. One exception is the material on the previously mentioned generalized Riemann integral, due independently to Jaroslav Kurzweil and Ralph Henstock. Section 8.1 closely follows the treatment laid out in Robert Bartle's article "Return to the Riemann Integral." In this paper, the author makes an italicized plea for teachers of mathematics to supplant Lebesgue's ubiquitous integral with the generalized Riemann integral. I hope that Professor Bartle will see its inclusion here as an inspired response to that request.

On a personal note, I welcome comments of any nature, and I will happily share any enlightening remarks—and any corrections—via a link on my webpage. The publication of this book comes nearly four years after the idea was first hatched. The long road to this point has required the steady support of many people but most notably that of my incredible wife, Katy. Amid the flurry of difficult decisions and hard work that go into a project of this size, the opportunity to dedicate this book to her comes as a pure and easy pleasure.

Middlebury, Vermont
August 2000

Stephen Abbott

Contents

Chapter 1

The Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

Toward the end of his distinguished career, the renowned British mathematician G.H. Hardy eloquently laid out a justification for a life of studying mathematics in *A Mathematician's Apology*, an essay first published in 1940. At the center of Hardy's defense is the thesis that mathematics is an aesthetic discipline. For Hardy, the applied mathematics of engineers and economists held little charm. "Real mathematics," as he referred to it, "must be justified as art if it can be justified at all."

To help make his point, Hardy includes two theorems from classical Greek mathematics, which, in his opinion, possess an elusive kind of beauty that, although difficult to define, is easy to recognize. The first of these results is Euclid's proof that there are an infinite number of prime numbers. The second result is the discovery, attributed to the school of Pythagoras from around 500 B.C., that $\sqrt{2}$ is irrational. It is this second theorem that demands our attention. (A course in number theory would focus on the first.) The argument uses only arithmetic, but its depth and importance cannot be overstated. As Hardy says, "[It] is a 'simple' theorem, simple both in idea and execution, but there is no doubt at all about [it being] of the highest class. [It] is as fresh and significant as when it was discovered—two thousand years have not written a wrinkle on [it]."

Theorem 1.1.1. *There is no rational number whose square is 2.*

Proof. A rational number is any number that can be expressed in the form p/q, where p and q are integers. Thus, what the theorem asserts is that no matter how p and q are chosen, it is never the case that $(p/q)^2 = 2$. The line of attack is indirect, using a type of argument referred to as a proof by contradiction. The idea is to assume that there *is* a rational number whose square is 2 and then proceed along logical lines until we reach a conclusion that is unacceptable. At this point, we will be forced to retrace our steps and reject the erroneous assumption that some rational number squared is equal to 2. In short, we will prove that the theorem is true by demonstrating that it cannot be false.

And so assume, for contradiction, that there exist integers p and q satisfying

$$\left(\frac{p}{q}\right)^2 = 2. \tag{1}$$

We may also assume that p and q have no common factor, because, if they had one, we could simply cancel it out and rewrite the fraction in lowest terms. Now, equation (1) implies

$$p^2 = 2q^2. \tag{2}$$

From this, we can see that the integer p^2 is an even number (it is divisible by 2), and hence p must be even as well because the square of an odd number is odd. This allows us to write $p = 2r$, where r is also an integer. If we substitute $2r$ for p in equation (2), then a little algebra yields the relationship

$$2r^2 = q^2.$$

But now the absurdity is at hand. This last equation implies that q^2 is even, and hence q must also be even. Thus, we have shown that p and q are both even (i.e., divisible by 2) when they were originally assumed to have no common factor. From this logical impasse, we can only conclude that equation (1) *cannot* hold for any integers p and q, and thus the theorem is proved. □

A component of Hardy's definition of beauty in a mathematical theorem is that the result have lasting and serious implications for a network of other mathematical ideas. In this case, the ideas under assault were the Greeks' understanding of the relationship between geometric *length* and arithmetic *number*. Prior to the preceding discovery, it was an assumed and commonly used fact that, given two line segments $\overline{AB}$ and $\overline{CD}$, it would always be possible to find a third line segment whose length divides evenly into the first two. In modern terminology, this is equivalent to asserting that the length of $\overline{CD}$ is a rational multiple of the length of $\overline{AB}$. Looking at the diagonal of a unit square (Fig. 1.1), it now followed (using the Pythagorean Theorem) that this was not always the case. Because the Pythagoreans implicitly interpreted number to mean rational number, they were forced to accept that number was a strictly weaker notion than length.

Rather than abandoning arithmetic in favor of geometry (as the Greeks seem to have done), our resolution to this limitation is to strengthen the concept of number by moving from the rational numbers to a larger number system. From a modern point of view, this should seem like a familiar and somewhat natural phenomenon. We begin with the *natural numbers*

$$\mathbf{N} = \{1, 2, 3, 4, 5, \dots\}.$$

The influential German mathematician Leopold Kronecker (1823–1891) once asserted that "The natural numbers are the work of God. All of the rest is

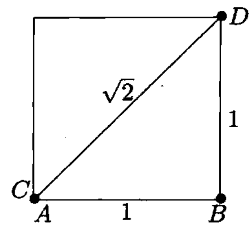

Figure 1.1: $\sqrt{2}$ EXISTS AS A GEOMETRIC LENGTH.

the work of mankind." Debating the validity of this claim is an interesting conversation for another time. For the moment, it at least provides us with a place to start. If we restrict our attention to the natural numbers **N**, then we can perform addition perfectly well, but we must extend our system to the *integers*

$$\mathbf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

if we want to have an additive identity (zero) and the additive inverses necessary to define subtraction. The next issue is multiplication and division. The number 1 acts as the multiplicative identity, but in order to define division we need to have multiplicative inverses. Thus, we extend our system again to the *rational numbers*

$$\mathbf{Q} = \left\{ \text{all fractions } \frac{p}{q} \text{ where } p \text{ and } q \text{ are integers with } q \neq 0 \right\}.$$

Taken together, the properties of **Q** discussed in the previous paragraph essentially make up the definition of what is called a *field*. More formally stated, a field is any set where addition and multiplication are well-defined operations that are commutative, associative, and obey the familiar distributive property $a(b+c) = ab + ac$. There must be an additive identity, and every element must have an additive inverse. Finally, there must be a multiplicative identity, and multiplicative inverses must exist for all nonzero elements of the field. Neither **Z** nor **N** is a field. The finite set $\{0, 1, 2, 3, 4\}$ is a field when addition and multiplication are computed modulo 5. This is not immediately obvious but makes an interesting exercise (Exercise 1.3.1).

The set **Q** also has a natural *order* defined on it. Given any two rational numbers r and s, exactly one of the following is true:

$$r < s, \quad r = s, \quad \text{or} \quad r > s.$$

This ordering is transitive in the sense that if $r < s$ and $s < t$, then $r < t$, so we are conveniently led to a mental picture of the rational numbers as being laid out from left to right along a number line. Unlike **Z**, there are no intervals of empty space. Given any two rational numbers $r < s$, the rational number

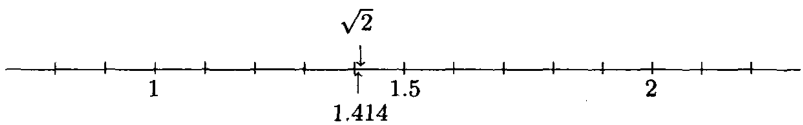

Figure 1.2: APPROXIMATING $\sqrt{2}$ WITH RATIONAL NUMBERS.

$(r+s)/2$ sits halfway in between, implying that the rational numbers are densely nestled together.

With the field properties of $\mathbf{Q}$ allowing us to safely carry out the algebraic operations of addition, subtraction, multiplication, and division, let's remind ourselves just what it is that $\mathbf{Q}$ is lacking. By Theorem 1.1.1, it is apparent that we cannot always take square roots. The problem, however, is actually more fundamental than this. Using only rational numbers, it is possible to *approximate* $\sqrt{2}$ quite well (Fig. 1.2). For instance, $1.414^2 = 1.999396$. By adding more decimal places to our approximation, we can get even closer to a value for $\sqrt{2}$, but, even so, we are now well aware that there is a "hole" in the rational number line where $\sqrt{2}$ ought to be. Of course, there are quite a few other holes—at $\sqrt{3}$ and $\sqrt{5}$, for example. Returning to the dilemma of the ancient Greek mathematicians, if we want every length along the number line to correspond to an actual number, then another extension to our number system is in order. Thus, to the chain $\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q}$ we append the *real numbers* $\mathbf{R}$.

The question of how to actually construct $\mathbf{R}$ from $\mathbf{Q}$ is rather complicated business. It is discussed in Section 1.3, and then again in more detail in Section 8.4. For the moment, it is not too inaccurate to say that $\mathbf{R}$ is obtained by filling in the gaps in $\mathbf{Q}$. Wherever there is a hole, a new *irrational* number is defined and placed into the ordering that already exists on $\mathbf{Q}$. The real numbers are then the union of these irrational numbers together with the more familiar rational ones. What properties does the set of irrational numbers have? How do the sets of rational and irrational numbers fit together? Is there a kind of symmetry between the rationals and the irrationals, or is there some sense in which we can argue that one type of real number is more common than the other? The one method we have seen so far for generating examples of irrational numbers is through square roots. Not too surprisingly, other roots such as $\sqrt[3]{2}$ or $\sqrt[5]{3}$ are most often irrational. Can all irrational numbers be expressed as algebraic combinations of nth roots and rational numbers, or are there still other irrational numbers beyond those of this form?

1.2 Some Preliminaries

The vocabulary necessary for the ensuing development comes from set theory and the theory of functions. This should be familiar territory, but a brief review of the terminology is probably a good idea, if only to establish some agreed-upon notation.

Sets

Intuitively speaking, a *set* is any collection of objects. These objects are referred to as the *elements* of the set. For our purposes, the sets in question will most often be sets of real numbers, although we will also encounter sets of functions and, on a few rare occasions, sets whose elements are other sets.

Given a set A, we write $x \in A$ if x (whatever it may be) is an element of A. If x is not an element of A, then we write $x \notin A$. Given two sets A and B, the *union* is written $A \cup B$ and is defined by asserting that

$$x \in A \cup B \text{ provided that } x \in A \text{ or } x \in B \text{ (or potentially both).}$$

The *intersection* $A \cap B$ is the set defined by the rule

$$x \in A \cap B \text{ provided } x \in A \text{ and } x \in B.$$

Example 1.2.1. (i) There are many acceptable ways to assert the contents of a set. In the previous section, the set of natural numbers was defined by listing the elements: $\mathbf{N} = \{1, 2, 3, \dots\}$.

(ii) Sets can also be described in words. For instance, we can define the set E to be the collection of even natural numbers.

(iii) Sometimes it is more efficient to provide a kind of rule or algorithm for determining the elements of a set. As an example, let

$$S = \{r \in \mathbf{Q} : r^2 < 2\}.$$

Read aloud, the definition of S says, "Let S be the set of all rational numbers whose squares are less than 2." It follows that $1 \in S$, $4/3 \in S$, but $3/2 \notin S$ because $9/4 \geq 2$.

Using the previously defined sets to illustrate the operations of intersection and union, we observe that

$$\mathbf{N} \cup E = \mathbf{N}, \quad \mathbf{N} \cap E = E, \quad \mathbf{N} \cap S = \{1\}, \text{ and } E \cap S = \emptyset.$$

The set $\emptyset$ is called the *empty set* and is understood to be the set that contains no elements. An equivalent statement would be to say that E and S are *disjoint.*

A word about the equality of two sets is in order (since we have just used the notion). The *inclusion* relationship $A \subseteq B$ or $B \supseteq A$ is used to indicate that every element of A is also an element of B. In this case, we say A is a *subset* of B, or B *contains* A. To assert that $A = B$ means that $A \subseteq B$ and $B \subseteq A$. Put another way, A and B have exactly the same elements.

Quite frequently in the upcoming chapters, we will want to apply the union and intersection operations to infinite collections of sets.

Example 1.2.2. Let

$$\begin{aligned} A_1 &= \mathbf{N} = \{1,2,3,\dots\}, \\ A_2 &= \{2,3,4,\dots\}, \\ A_3 &= \{3,4,5,\dots\}, \end{aligned}$$

and, in general, for each $n \in \mathbf{N}$, define the set

$$A_n = \{n, n+1, n+2, \dots\}.$$

The result is a nested chain of sets

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \cdots,$$

where each successive set is a subset of all the previous ones. Notationally,

$$\bigcup_{n=1}^{\infty} A_n, \quad \bigcup_{n \in \mathbf{N}} A_n, \quad \text{or} \quad A_1 \cup A_2 \cup A_3 \cup \cdots$$

are all equivalent ways to indicate the set whose elements consist of any element that appears in at least one particular A_n. Because of the nested property of this particular collection of sets, it is not too hard to see that

$$\bigcup_{n=1}^{\infty} A_n = A_1.$$

The notion of intersection has the same kind of natural extension to infinite collections of sets. For this example, we have

$$\bigcap_{n=1}^{\infty} A_n = \emptyset.$$

Let's be sure we understand why this is the case. Suppose we had some natural number m that we thought might actually satisfy $m \in \bigcap_{n=1}^{\infty} A_n$. What this would mean is that $m \in A_n$ for *every* A_n in our collection of sets. Because m is not an element of A_{m+1}, no such m exists and the intersection is empty.

As mentioned, most of the sets we encounter will be sets of real numbers. Given $A \subseteq \mathbf{R}$, the *complement* of A, written A^c, refers to the set of all elements of $\mathbf{R}$ not in A. Thus, for $A \subseteq \mathbf{R}$,

$$A^c = \{x \in \mathbf{R} : x \notin A\}.$$

A few times in our work to come, we will refer to De Morgan's Laws, which state that

$$(A \cap B)^c = A^c \cup B^c \quad \text{and} \quad (A \cup B)^c = A^c \cap B^c.$$

Proofs of these statements are discussed in Exercise 1.2.3.