William P.Ziemer

Weakly Differentiable Functions

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Sobolev Spaces and Functions of Bounded Variation

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To Suzanne

Preface

The term "weakly differentiable functions" in the title refers to those integrable functions defined on an open subset of R^n whose partial derivatives in the sense of distributions are either L^p functions or (signed) measures with finite total variation. The former class of functions comprises what is now known as Sobolev spaces, though its origin, traceable to the early 1900s, predates the contributions by Sobolev. Both classes of functions, Sobolev spaces and the space of functions of bounded variation (BV functions), have undergone considerable development during the past 20 years. From this development a rather complete theory has emerged and thus has provided the main impetus for the writing of this book. Since these classes of functions play a significant role in many fields, such as approximation theory, calculus of variations, partial differential equations, and non-linear potential theory, it is hoped that this monograph will be of assistance to a wide range of graduate students and researchers in these and perhaps other related areas. Some of the material in Chapters 1–4 has been presented in a graduate course at Indiana University during the 1987–88 academic year, and I am indebted to the students and colleagues in attendance for their helpful comments and suggestions.

The major thrust of this book is the analysis of pointwise behavior of Sobolev and BV functions. I have not attempted to develop Sobolev spaces of fractional order which can be described in terms of Bessel potentials, since this would require an effort beyond the scope of this book. Instead, I concentrate on the analysis of spaces of integer order which is largely accessible through real variable techniques, but does not totally exclude the use of Bessel potentials. Indeed, the investigation of pointwise behavior requires an analysis of certain exceptional sets and they can be conveniently described in terms of elementary aspects of Bessel capacity.

The only prerequisite for the present volume is a standard graduate course in real analysis, drawing especially from Lebesgue point theory and measure theory. The material is organized in the following manner. Chapter 1 is devoted to a review of those topics in real analysis that are needed in the sequel. Included here is a brief overview of Lebesgue measure, L^p spaces, Hausdorff measure, and Schwartz distributions. Also included are sections on covering theorems and Lorentz spaces—the latter being necessary for a treatment of Sobolev inequalities in the case of critical indices. Chapter 2 develops the basic properties of Sobolev spaces such as equivalent formulations of Sobolev functions and their behavior under the operations of truncation, composition, and change of variables. Also included is a proof of the Sobolev inequality in its simplest form and the related Rellich-Kondrachov Compactness Theorem. Alternate proofs of the Sobolev inequality are given, including the one which relates it to the isoperimetric inequality and provides the best constant. Limiting cases of the Sobolev inequality are discussed in the context of Lorentz spaces.

The remaining chapters are central to the book. Chapter 3 develops the analysis of pointwise behavior of Sobolev functions. This includes a discussion of the continuity properties of functions with first derivatives in L^p in terms of Lebesgue points, approximate continuity, and fine continuity, as well as an analysis of differentiability properties of higher order Sobolev functions by means of L^p-derivatives. Here lies the foundation for more delicate results, such as the comparison of L^p-derivatives and distributional derivatives, and a result which provides an approximation for Sobolev functions by smooth functions (in norm) that agree with the given function everywhere except on sets whose complements have small capacity.

Chapter 4 develops an idea due to Norman Meyers. He observed that the usual indirect proof of the Poincaré inequality could be used to establish a Poincaré-type inequality in an abstract setting. By appropriately interpreting this inequality in various contexts, it yields virtually all known inequalities of this genre. This general inequality contains a term which involves an element of the dual of a Sobolev space. For many applications, this term is taken as a measure; it therefore is of interest to know precisely the class of measures contained in the dual of a given Sobolev space. Fortunately, the Hedberg–Wolff theorem provides a characterization of such measures.

The last chapter provides an analysis of the pointwise behavior of BV functions in a manner that runs parallel to the development of Lebesgue point theory for Sobolev functions in Chapter 3. While the Lebesgue point theory for Sobolev functions is relatively easy to penetrate, the corresponding development for BV functions is much more demanding. The intricate nature of BV functions requires a more involved exposition than does Sobolev functions, but at the same time reveals a rich and beautiful structure which has its foundations in geometric measure theory. After the structure of BV functions has been developed, Chapter 5 returns to the analysis of Poincaré inequalities for BV functions in the spirit developed for Sobolev functions, which includes a characterization of measures that belong to the dual of BV.

In order to place the text in better perspective, each chapter is concluded with a section on historical notes which includes references to all important and relatively new results. In addition to cited works, the Bibliography contains many other references related to the material in the text. Bibliographical references are abbreviated in square brackets, such as [DL]. Equation numbers appear in parentheses; theorems, lemmas, corollaries,and remarks are numbered as $a.b.c$ where b refers to section b in chapter *a*, and section *a.b* refers to section *b* in chapter *a*.

I wish to thank David Adams, Robert Glassey, Tero Kilpeläinen, Christoph Neugebauer, Edward Stredulinsky, Tevan Trent, and William K. Ziemer for having critically read parts of the manuscript and supplied many helpful suggestions and corrections.

WILLIAM P. ZIEMER

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Preliminaries

Beyond the topics usually found in basic real analysis, virtually all of the material found in this work is self-contained. In particular, most of the information contained in this chapter will be well-known by the reader and therefore no attempt has been made to make a complete and thorough presentation. Rather, we merely introduce notation and develop a few concepts that will be needed in the sequel.

1.1 Notation

Throughout, the symbol Ω will generally denote an open set in Euclidean space R^n and $\emptyset$ will designate the empty set. Points in R^n are denoted by $x = (x_1, \ldots, x_n)$, where $x_1 \in R^1$, $1 \le i \le n$. If $x, y \in R^n$, the *inner product* of x and y is

$$x \cdot y = \sum_{i=1}^{n} x_i y_i$$

and the *norm* of x is

$$|x| = (x \cdot x)^{1/2}.$$

If $u: \Omega \to R^1$ is a function defined on Ω, the *support* of u is defined by

$$\operatorname{spt} u = \Omega \cap \overline{\{x: u(x) \neq 0\}},$$

where the closure of a set $S \subset R^n$ is denoted by $\overline{S}$. If $S \subset \Omega$, $\overline{S}$ compact and also $\overline{S} \subset \Omega$, we shall write $S \subset\subset \Omega$. The *boundary* of a set S is defined by

$$\partial S = \overline{S} \cap \overline{(R^n - S)}.$$

For $E \subset R^n$ and $x \in R^n$, the distance from x to E is

$$d(x, E) = \inf\{|x - y| : y \in E\}.$$

It is a simple exercise (see Exercise 1.1) to show that

$$|d(x, E) - d(y, E)| \le |x - y|$$

whenever $x, y \in R^n$. The diameter of a set $E \subset R^n$ is defined by

$$\operatorname{diam}(E) = \sup\{|x - y| : x, y \in E\},$$

and the characteristic function E is denoted by χ_E. The symbol

$$B(x,r) = \{y : |x-y| < r\}$$

denotes the open ball with center x, radius r and

$$\overline{B}(x,r) = \{y : |x-y| \leq r\}$$

will stand for the closed ball. We will use $\alpha(n)$ to denote the volume of the ball of radius 1 in R^n. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of non-negative integers, α is called a *multi-index* and the *length* of α is

$$|\alpha| = \sum_{i=1}^{n} \alpha_i.$$

If $x = (x_1, \ldots, x_n) \in R^n$, we will let

$$x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

and $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$. The *partial derivative operators* are denoted by $D_i = \partial/\partial x_i$ for $1 \leq i \leq n$, and the higher order derivatives by

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

The *gradient* of a real-valued function u is denoted by

$$Du(x) = (D_1 u(x), \ldots, D_n u(x)).$$

If k is a non-negative integer, we will sometimes use $D^k u$ to denote the vector $D^k u = \{D^\alpha u\}_{|\alpha|=k}$.

We denote by $C^0(\Omega)$ the space of continuous functions on Ω. More generally, if k is a non-negative integer, possibly ∞, let

$$C^k(\Omega) = \{u : u: \Omega \to R^1, D^\alpha u \in C^0(\Omega), \quad 0 \leq |\alpha| \leq k\},$$

$$C_0^k(\Omega) = C^k(\Omega) \cap \{u : \text{spt } u \text{ compact, spt } u \subset \Omega\},$$

and

$$C^k(\overline{\Omega}) = C^k(\Omega) \cap \{u : D^\alpha u \text{ has a continuous extension to } \overline{\Omega}, 0 \leq |\alpha| \leq k\}.$$

Since Ω is open, a function $u \in C^k(\Omega)$ need not be bounded on Ω. However, if u is bounded and uniformly continuous on Ω, then u can be uniquely extended to a continuous function on $\overline{\Omega}$. We will use $C^k(\Omega; R^m)$ to denote the class of functions $u: \Omega \to R^m$ defined on Ω whose coordinate functions belong to $C^k(\Omega)$. Similar notation is used for other function spaces whose elements are vector-valued.