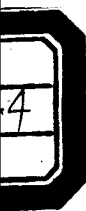


William P. Ziemer

# **Weakly Differentiable Functions**

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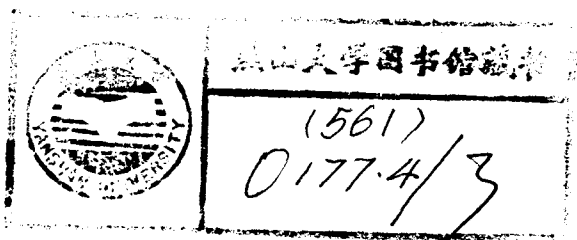
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William P. Ziemer

# Weakly Differentiable Functions

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Bounded Variation*



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continued after Index

To Suzanne

# Preface

The term "weakly differentiable functions" in the title refers to those integrable functions defined on an open subset of  $R^n$  whose partial derivatives in the sense of distributions are either  $L^p$  functions or (signed) measures with finite total variation. The former class of functions comprises what is now known as Sobolev spaces, though its origin, traceable to the early 1900s, predates the contributions by Sobolev. Both classes of functions, Sobolev spaces and the space of functions of bounded variation (BV functions), have undergone considerable development during the past 20 years. From this development a rather complete theory has emerged and thus has provided the main impetus for the writing of this book. Since these classes of functions play a significant role in many fields, such as approximation theory, calculus of variations, partial differential equations, and non-linear potential theory, it is hoped that this monograph will be of assistance to a wide range of graduate students and researchers in these and perhaps other related areas. Some of the material in Chapters 1-4 has been presented in a graduate course at Indiana University during the 1987-88 academic year, and I am indebted to the students and colleagues in attendance for their helpful comments and suggestions.

The major thrust of this book is the analysis of pointwise behavior of Sobolev and BV functions. I have not attempted to develop Sobolev spaces of fractional order which can be described in terms of Bessel potentials, since this would require an effort beyond the scope of this book. Instead, I concentrate on the analysis of spaces of integer order which is largely accessible through real variable techniques, but does not totally exclude the use of Bessel potentials. Indeed, the investigation of pointwise behavior requires an analysis of certain exceptional sets and they can be conveniently described in terms of elementary aspects of Bessel capacity.

The only prerequisite for the present volume is a standard graduate course in real analysis, drawing especially from Lebesgue point theory and measure theory. The material is organized in the following manner. Chapter 1 is devoted to a review of those topics in real analysis that are needed in the sequel. Included here is a brief overview of Lebesgue measure,  $L^p$  spaces, Hausdorff measure, and Schwartz distributions. Also included are sections on covering theorems and Lorentz spaces—the latter being necessary for a treatment of Sobolev inequalities in the case of critical indices. Chapter 2 develops the basic properties of Sobolev spaces such as equivalent formulations of Sobolev functions and their behavior under the opera-

tions of truncation, composition, and change of variables. Also included is a proof of the Sobolev inequality in its simplest form and the related Rellich-Kondrachov Compactness Theorem. Alternate proofs of the Sobolev inequality are given, including the one which relates it to the isoperimetric inequality and provides the best constant. Limiting cases of the Sobolev inequality are discussed in the context of Lorentz spaces.

The remaining chapters are central to the book. Chapter 3 develops the analysis of pointwise behavior of Sobolev functions. This includes a discussion of the continuity properties of functions with first derivatives in  $L^p$  in terms of Lebesgue points, approximate continuity, and fine continuity, as well as an analysis of differentiability properties of higher order Sobolev functions by means of  $L^p$ -derivatives. Here lies the foundation for more delicate results, such as the comparison of  $L^p$ -derivatives and distributional derivatives, and a result which provides an approximation for Sobolev functions by smooth functions (in norm) that agree with the given function everywhere except on sets whose complements have small capacity.

Chapter 4 develops an idea due to Norman Meyers. He observed that the usual indirect proof of the Poincaré inequality could be used to establish a Poincaré-type inequality in an abstract setting. By appropriately interpreting this inequality in various contexts, it yields virtually all known inequalities of this genre. This general inequality contains a term which involves an element of the dual of a Sobolev space. For many applications, this term is taken as a measure; it therefore is of interest to know precisely the class of measures contained in the dual of a given Sobolev space. Fortunately, the Hedberg-Wolff theorem provides a characterization of such measures.

The last chapter provides an analysis of the pointwise behavior of BV functions in a manner that runs parallel to the development of Lebesgue point theory for Sobolev functions in Chapter 3. While the Lebesgue point theory for Sobolev functions is relatively easy to penetrate, the corresponding development for BV functions is much more demanding. The intricate nature of BV functions requires a more involved exposition than does Sobolev functions, but at the same time reveals a rich and beautiful structure which has its foundations in geometric measure theory. After the structure of BV functions has been developed, Chapter 5 returns to the analysis of Poincaré inequalities for BV functions in the spirit developed for Sobolev functions, which includes a characterization of measures that belong to the dual of BV.

In order to place the text in better perspective, each chapter is concluded with a section on historical notes which includes references to all important and relatively new results. In addition to cited works, the Bibliography contains many other references related to the material in the text. Bibliographical references are abbreviated in square brackets, such as [DL]. Equation numbers appear in parentheses; theorems, lemmas, corollaries, and remarks are numbered as  $a.b.c$  where  $b$  refers to section  $b$  in chapter



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$a$ , and section  $a.b$  refers to section  $b$  in chapter  $a$ .

I wish to thank David Adams, Robert Glassey, Tero Kilpeläinen, Christoph Neugebauer, Edward Stredulinsky, Tevan Trent, and William K. Ziemer for having critically read parts of the manuscript and supplied many helpful suggestions and corrections.

WILLIAM P. ZIEMER

# Contents

Preface	vii
1 Preliminaries	1
1.1 Notation	1
Inner product of vectors	
Support of a function	
Boundary of a set	
Distance from a point to a set	
Characteristic function of a set	
Multi-indices	
Partial derivative operators	
Function spaces—continuous, Hölder continuous,	
Hölder continuous derivatives	
1.2 Measures on $R^n$	3
Lebesgue measurable sets	
Lebesgue measurability of Borel sets	
Suslin sets	
1.3 Covering Theorems	7
Hausdorff maximal principle	
General covering theorem	
Vitali covering theorem	
Covering lemma, with $n$ -balls whose radii vary in	
Lipschitzian way	
Besicovitch covering lemma	
Besicovitch differentiation theorem	
1.4 Hausdorff Measure	15
Equivalence of Hausdorff and Lebesgue measures	
Hausdorff dimension	
1.5 $L^p$ -Spaces	18
Integration of a function via its distribution	
function	
Young's inequality	
Hölder's and Jensen's inequality	
1.6 Regularization	21
$L^p$ -spaces and regularization	

1.7	Distributions	23
	Functions and measures, as distributions	
	Positive distributions	
	Distributions determined by their local behavior	
	Convolution of distributions	
	Differentiation of distributions	
1.8	Lorentz Spaces	26
	Non-increasing rearrangement of a function	
	Elementary properties of rearranged functions	
	Lorentz spaces	
	O'Neil's inequality, for rearranged functions	
	Equivalence of $L^p$ -norm and $(p, p)$ -norm	
	Hardy's inequality	
	Inclusion relations of Lorentz spaces	
	Exercises	37
	Historical Notes	39
2	Sobolev Spaces and Their Basic Properties	42
2.1	Weak Derivatives	42
	Sobolev spaces	
	Absolute continuity on lines	
	$L^p$ -norm of difference quotients	
	Truncation of Sobolev functions	
	Composition of Sobolev functions	
2.2	Change of Variables for Sobolev Functions	49
	Rademacher's theorem	
	Bi-Lipschitzian change of variables	
2.3	Approximation of Sobolev Functions by Smooth Functions	53
	Partition of unity	
	Smooth functions are dense in $W^{k,p}$	
2.4	Sobolev Inequalities	55
	Sobolev's inequality	
2.5	The Rellich-Kondrachov Compactness Theorem	61
	Extension domains	
2.6	Bessel Potentials and Capacity	64
	Riesz and Bessel kernels	
	Bessel potentials	
	Bessel capacity	
	Basic properties of Bessel capacity	
	Capacitability of Suslin sets	
	Minimax theorem and alternate formulation of Bessel capacity	

	Metric properties of Bessel capacity	
2.7	The Best Constant in the Sobolev Inequality	76
	Co-area formula	
	Sobolev's inequality and isoperimetric inequality	
2.8	Alternate Proofs of the Fundamental Inequalities	83
	Hardy-Littlewood-Wiener maximal theorem	
	Sobolev's inequality for Riesz potentials	
2.9	Limiting Cases of the Sobolev Inequality	88
	The case $kp = n$ by infinite series	
	The best constant in the case $kp = n$	
	An $L^\infty$ -bound in the limiting case	
2.10	Lorentz Spaces, A Slight Improvement	96
	Young's inequality in the context of Lorentz spaces	
	Sobolev's inequality in Lorentz spaces	
	The limiting case	
	Exercises	103
	Historical Notes	108
3	Pointwise Behavior of Sobolev Functions	112
3.1	Limits of Integral Averages of Sobolev Functions	112
	Limiting values of integral averages except for capacity null set	
3.2	Densities of Measures	116
3.3	Lebesgue Points for Sobolev Functions	118
	Existence of Lebesgue points except for capacity null set	
	Approximate continuity	
	Fine continuity everywhere except for capacity null set	
3.4	$L^p$ -Derivatives for Sobolev Functions	126
	Existence of Taylor expansions $L^p$	
3.5	Properties of $L^p$ -Derivatives	130
	The spaces $T^k, t^k, T^{k,p}, t^{k,p}$	
	The implication of a function being in $T^{k,p}$ at all points of a closed set	
3.6	An $L^p$ -Version of the Whitney Extension Theorem	136
	Existence of a $C^\infty$ function comparable to the distance function to a closed set	
	The Whitney extension theorem for functions in $T^{k,p}$ and $t^{k,p}$	
3.7	An Observation on Differentiation	142
3.8	Rademacher's Theorem in the $L^p$ -Context	145
	A function in $T^{k,p}$ everywhere implies it is in $t^{k,p}$ almost everywhere	

3.9	The Implications of Pointwise Differentiability Comparison of $L^p$ -derivatives and distributional derivatives If $u \in t^{k,p}(x)$ for every $x$ , and if the $L^p$ -derivatives are in $L^p$ , then $u \in W^{k,p}$	146
3.10	A Lusin-Type Approximation for Sobolev Functions Integral averages of Sobolev functions are uniformly close to their limits on the complement of sets of small capacity Existence of smooth functions that agree with Sobolev functions on the complement of sets of small capacity	153
3.11	The Main Approximation Existence of smooth functions that agree with Sobolev functions on the complement of sets of small capacity and are close in norm	159
	Exercises	168
	Historical Notes	175
4	Poincaré Inequalities—A Unified Approach	177
4.1	Inequalities in a General Setting An abstract version of the Poincaré inequality	178
4.2	Applications to Sobolev Spaces An interpolation inequality	182
4.3	The Dual of $W^{m,p}(\Omega)$ The representation of $(W_0^{m,p}(\Omega))^*$	185
4.4	Some Measures in $(W_0^{m,p}(\Omega))^*$ Poincaré inequalities derived from the abstract version by identifying Lebesgue and Hausdorff measure with elements in $(W^{m,p}(\Omega))^*$ The trace of Sobolev functions on the boundary of Lipschitz domains Poincaré inequalities involving the trace of a Sobolev function	188
4.5	Poincaré Inequalities Inequalities involving the capacity of the set on which a function vanishes	193
4.6	Another Version of Poincaré's Inequality An inequality involving dependence on the set on which the function vanishes, not merely on its capacity	196
4.7	More Measures in $(W^{m,p}(\Omega))^*$ Sobolev's inequality for Riesz potentials involving	198

	measures other than Lebesgue measure	
	Characterization of measures in $(W^{m,p}(R^n))^*$	
4.8	Other Inequalities Involving Measures in $(W^{k,p})^*$	207
	Inequalities involving the restriction of Hausdorff measure to lower dimensional manifolds	
4.9	The Case $p = 1$	209
	Inequalities involving the $L^1$ -norm of the gradient	
	Exercises	214
	Historical Notes	217
5	Functions of Bounded Variation	220
5.1	Definitions	220
	Definition of BV functions	
	The total variation measure $\ Du\ $	
5.2	Elementary Properties of BV Functions	222
	Lower semicontinuity of the total variation measure	
	A condition ensuring continuity of the total variation measure	
5.3	Regularization of BV Functions	224
	Regularization does not increase the BV norm	
	Approximation of BV functions by smooth functions	
	Compactness in $L^1$ of the unit ball in BV	
5.4	Sets of Finite Perimeter	228
	Definition of sets of finite perimeter	
	The perimeter of domains with smooth boundaries	
	Isoperimetric and relative isoperimetric inequality for sets of finite perimeter	
5.5	The Generalized Exterior Normal	233
	A preliminary version of the Gauss-Green theorem	
	Density results at points of the reduced boundary	
5.6	Tangential Properties of the Reduced Boundary and the Measure-Theoretic Normal	237
	Blow-up at a point of the reduced boundary	
	The measure-theoretic normal	
	The reduced boundary is contained in the measure-theoretic boundary	
	A lower bound for the density of $\ D\chi_E\ $	
	Hausdorff measure restricted to the reduced boundary is bounded above by $\ D\chi_E\ $	
5.7	Rectifiability of the Reduced Boundary	243
	Countably $(n-1)$ -rectifiable sets	
	Countable $(n-1)$ -rectifiability of the measure-theoretic boundary	

5.8	The Gauss–Green Theorem	246
	The equivalence of the restriction of Hausdorff measure to the measure-theoretic boundary and $\ DX_E\ $	
	The Gauss–Green theorem for sets of finite perimeter	
5.9	Pointwise Behavior of BV Functions	249
	Upper and lower approximate limits	
	The Boxing inequality	
	The set of approximate jump discontinuities	
5.10	The Trace of a BV Function	255
	The bounded extension of BV functions	
	Trace of a BV function defined in terms of the upper and lower approximate limits of the extended function	
	The integrability of the trace over the measure-theoretic boundary	
5.11	Sobolev-Type Inequalities for BV Functions	260
	Inequalities involving elements in $(BV(\Omega))^*$	
5.12	Inequalities Involving Capacity	262
	Characterization of measure in $(BV(\Omega))^*$	
	Poincaré inequality for BV functions	
5.13	Generalizations to the Case $p > 1$	270
5.14	Trace Defined in Terms of Integral Averages	272
	Exercises	277
	Historical Notes	280
	Bibliography	283
	List of Symbols	297
	Index	303

# 1

## Preliminaries

Beyond the topics usually found in basic real analysis, virtually all of the material found in this work is self-contained. In particular, most of the information contained in this chapter will be well-known by the reader and therefore no attempt has been made to make a complete and thorough presentation. Rather, we merely introduce notation and develop a few concepts that will be needed in the sequel.

### 1.1 Notation

Throughout, the symbol  $\Omega$  will generally denote an open set in Euclidean space  $R^n$  and  $\emptyset$  will designate the empty set. Points in  $R^n$  are denoted by  $x = (x_1, \dots, x_n)$ , where  $x_i \in R^1$ ,  $1 \leq i \leq n$ . If  $x, y \in R^n$ , the *inner product* of  $x$  and  $y$  is

$$x \cdot y = \sum_{i=1}^n x_i y_i$$

and the *norm* of  $x$  is

$$|x| = (x \cdot x)^{1/2}.$$

If  $u: \Omega \rightarrow R^1$  is a function defined on  $\Omega$ , the *support* of  $u$  is defined by

$$\text{spt } u = \Omega \cap \overline{\{x: u(x) \neq 0\}},$$

where the closure of a set  $S \subset R^n$  is denoted by  $\bar{S}$ . If  $S \subset \Omega$ ,  $\bar{S}$  compact and also  $\bar{S} \subset \Omega$ , we shall write  $S \subset\subset \Omega$ . The *boundary* of a set  $S$  is defined by

$$\partial S = \bar{S} \cap \overline{(R^n - S)}.$$

For  $E \subset R^n$  and  $x \in R^n$ , the distance from  $x$  to  $E$  is

$$d(x, E) = \inf\{|x - y| : y \in E\}.$$

It is a simple exercise (see Exercise 1.1) to show that

$$|d(x, E) - d(y, E)| \leq |x - y|$$

whenever  $x, y \in R^n$ . The diameter of a set  $E \subset R^n$  is defined by

$$\text{diam}(E) = \sup\{|x - y| : x, y \in E\},$$



and the characteristic function  $E$  is denoted by  $\chi_E$ . The symbol

$$B(x, r) = \{y : |x - y| < r\}$$

denotes the open ball with center  $x$ , radius  $r$  and

$$\bar{B}(x, r) = \{y : |x - y| \leq r\}$$

will stand for the closed ball. We will use  $\alpha(n)$  to denote the volume of the ball of radius 1 in  $R^n$ . If  $\alpha = (\alpha_1, \dots, \alpha_n)$  is an  $n$ -tuple of non-negative integers,  $\alpha$  is called a *multi-index* and the *length* of  $\alpha$  is

$$|\alpha| = \sum_{i=1}^n \alpha_i.$$

If  $x = (x_1, \dots, x_n) \in R^n$ , we will let

$$x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

and  $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$ . The *partial derivative operators* are denoted by  $D_i = \partial/\partial x_i$  for  $1 \leq i \leq n$ , and the higher order derivatives by

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

The *gradient* of a real-valued function  $u$  is denoted by

$$Du(x) = (D_1 u(x), \dots, D_n u(x)).$$

If  $k$  is a non-negative integer, we will sometimes use  $D^k u$  to denote the vector  $D^k u = \{D^\alpha u\}_{|\alpha|=k}$ .

We denote by  $C^0(\Omega)$  the space of continuous functions on  $\Omega$ . More generally, if  $k$  is a non-negative integer, possibly  $\infty$ , let

$$C^k(\Omega) = \{u : u : \Omega \rightarrow R^1, D^\alpha u \in C^0(\Omega), 0 \leq |\alpha| \leq k\},$$

$$C_0^k(\Omega) = C^k(\Omega) \cap \{u : \text{spt } u \text{ compact, spt } u \subset \Omega\},$$

and

$$C^k(\bar{\Omega}) = C^k(\Omega) \cap \{u : D^\alpha u \text{ has a continuous extension to } \bar{\Omega}, 0 \leq |\alpha| \leq k\}.$$

Since  $\Omega$  is open, a function  $u \in C^k(\Omega)$  need not be bounded on  $\Omega$ . However, if  $u$  is bounded and uniformly continuous on  $\Omega$ , then  $u$  can be uniquely extended to a continuous function on  $\bar{\Omega}$ . We will use  $C^k(\Omega; R^m)$  to denote the class of functions  $u : \Omega \rightarrow R^m$  defined on  $\Omega$  whose coordinate functions belong to  $C^k(\Omega)$ . Similar notation is used for other function spaces whose elements are vector-valued.