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Advances in Two-Dimensional Homotopy and Combinatorial Group Theory

Edited by

Wolfgang Metzler and Stephan Rosebrock



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Advances in Two-Dimensional Homotopy and Combinatorial Group Theory

Edited by Wolfgang Metzler and Stephan Rosebrock

This volume presents the current state of knowledge in all aspects of two-dimensional homotopy theory. Building on the foundations laid a quarter of a century ago in the volume *Two-dimensional Homotopy and Combinatorial Group Theory* (LMS 197), the editors here bring together much remarkable progress that has been obtained in the intervening years. And while the fundamental open questions, such as the Andrews–Curtis Conjecture and the Whitehead asphericity problem remain to be (fully) solved, this book will provide both students and experts with an overview of the state of the art and work in progress. Ample references are included to the LMS 197 volume, as well as a comprehensive bibliography bringing matters entirely up to date.

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Advances in Two-Dimensional Homotopy and Combinatorial Group Theory

Edited by

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Editors' Preface

Our book on "Two-dimensional Homotopy and Combinatorial Group Theory (LMS 197)" appeared 24 years ago. Since the turn of the millennium we were encouraged to publish in a similar style the (remarkable) progress that has been obtained in the meantime. On the other hand, none of the fundamental open questions like the Andrews-Curtis Conjecture and the Whitehead-Asphericity problem have been (fully) solved, and almost no prognosis is possible about when it will finally happen. We hope that mathematicians, who are in danger of leaving the subject because of this perspective, will be kept active by this sequel publication; and new ones will be gained for our important field.

Again, we mix surveys and original material in the chapters, which can be read by students and experts alike. Ongoing research is mentioned as work in progress. There are ample references to the LMS 197 volume which will be cited as [I] including the bibliography and the index, where new publications and new notions are listed.

Since the publication of LMS 197 there have been workshops of the authors at least every fifth year. We met in Germany, Italy, Russia and the USA, and the fruits of these meetings are the basis of this book. During a meeting close to Naumburg (Saale) we made the decision to start what you can read now.

A mathematically energetic member of the team of authors was Nancy Waller. But when the actual writing had to start, she unexpectedly died. Her preprint [BeWa13] together with F. Rudolf Beyl could not be finished, and in particular it was impossible without her to write a planned chapter on the *geometric realization problem for algebraic 2-complexes* at the same time. Hence "Nancy's Toy", which she hoped that would result in a counterexample, has been left to the future. What can be said at present, is collected in an appendix to chapter I by F. Rudolf Beyl and Jens Harlander. This book can be considered as serving for Nancy Waller's memory.

As for the LMS 197 volume, all chapters are refereed within the team of au-

thors and by an external referee who had the option of remaining anonymous. We are grateful for their valuable service. Those who gave us permission to be named openly are Laura Biroth (Mainz), Martin Bridson (Oxford), Cameron Gordon (Austin, TX), Paul Latiolais (Portland, OR), Peter Linnell (Blacksburg, VA), Wajid Mannan (Brighton), and Timo deWolff (College Station, TX). We also thank our former students for their help to prepare this book, some of whom are mentioned in the chapters and in the bibliography. A special thank you goes to Wesley Browning, who allowed us to reproduce his note [Bro76], which grew out of correspondence with Joan Birman and which should not be forgotten.

Last, but not least, we again thank Roger Astley and his team, who gave encouragement and continuous advice as for LMS 197.

July 2017

Wolfgang Metzler and Stephan Rosebrock

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1

A Survey of recent Progress on some Problems in 2-dimensional Topology

Jens Harlander

1.1 Introduction

A primary theme in algebraic topology is the characterization of the properties of a space or group in homological terms. Specific examples are Wall's questions concerning finiteness properties of spaces, or the Eilenberg-Ganea problem that compares the cohomological dimension of a group with its geometric dimension. Relevant to both examples is the question: can a partial resolution of length n be geometrically realized?

Any partial resolution of length n is chain homotopically equivalent to one of length n with geometric $(n-1)$ -skeleton. For $n \geq 3$, the $(n-1)$ -skeleton of the universal covering of a classifying space is simply connected, and this fact can be used to obtain a geometric realization of the original partial resolution. For $n = 2$, however, simple connectivity fails and the realization question translates into difficult questions concerning presentations of groups. These questions are the topic of this chapter.

We will introduce the *geometric realization problem* in dimension 2, the *relation lifting problem* and the *relation gap problem*. The first in this list of problems is the most fundamental, while the second and third can be stated purely in terms of combinatorial group theory.

The final section in this chapter is an appendix jointly written by F. Rudolf Beyl and the author. It gives a brief overview of the state of the geometric realization problem for finite groups.

This first chapter also serves as an introduction to two later chapters: Chapter 6 is a detailed account of the relation gap problem, and Chapter 7 discusses the relation gap problem in the context of free products.

1.2 Basic definitions

Let G be a group and $\mathbf{g}$ be a generating set for G . Let $\mathbf{x}$ be a set of letters and $\phi: \mathbf{x} \rightarrow \mathbf{g}$ be a bijection. This bijection extends to a group epimorphism

$$\phi: F(\mathbf{x}) \longrightarrow G,$$

from the free group $F = F(\mathbf{x})$ onto G . The kernel N of ϕ is called the *relation group* associated with the generating set $\mathbf{g}$ of G . The conjugation action of F on N provides a $\mathbb{Z}G$ -module structure on $N_{ab} = N/[N, N]$: if $g \in G$ and $r \in N$, then the action of G on $N/[N, N]$ is well-defined by $g \cdot r[N, N] = frf^{-1}[N, N]$, with any $f \in F$ such that $g = \phi(f)$. This module is called the *relation module* associated with the generating set $\mathbf{g}$ of G . If we choose a set $\mathbf{r}$ of normal generators of N , then

$$P = \langle \mathbf{x} \mid \mathbf{r} \rangle$$

is called a *presentation* of the group G . By a slight abuse of notation we also refer to

$$1 \longrightarrow N \longrightarrow F(\mathbf{x}) \xrightarrow{\phi} G \longrightarrow 1,$$

or just the quotient F/N , as a presentation of the group G . A presentation is called *finite* if both $\mathbf{x}$ and $\mathbf{r}$ are finite sets.

The *standard 2-complex* $K(P)$ associated with P is constructed as follows: $K(P)$ has a single 0-cell, 1-cells in one-to-one correspondence with the generators in P , and 2-cells in one-to-one correspondence with the relators $r \in \mathbf{r}$ in P . A word in the generators and their inverses defines a path in $K(P)$. The 2-cell in $K(P)$ corresponding to a relator r is attached to the 1-skeleton via the path corresponding to the word r in the generators. For convenience we will simply refer to the x_i as 1-cells of $K(P)$ and the $r \in \mathbf{r}$ as 2-cells of $K(P)$.

We will next describe the universal covering $\tilde{K}(P)$ of $K(P)$. The complex $\tilde{K}(P)$ has 0-cells in one-to-one correspondence with the elements of G , and oriented edges $g\tilde{x}$, $g \in G$, $x \in \mathbf{x}$. The initial vertex of $g\tilde{x}$ is g , its terminal vertex is $g\phi(x)$. We call edges of the form $g\tilde{x}$ edges with label x . Note that at a vertex g we have a unique outgoing edge with label x (the edge $g\tilde{x}$) and a unique incoming edge with label x (the edge $g\phi(x)^{-1}\tilde{x}$). Thus a path in $K(P)$, which corresponds to a word w in the generators and their inverses, can be uniquely lifted to a path $g\tilde{w}$, starting at g and ending at $g\phi(w)$. Reading the labels along the path $g\tilde{w}$ reads the word w . For every $g \in G$ and $r \in \mathbf{r}$ we have a 2-cell in $\tilde{K}(P)$ attached along the closed path $g\tilde{r}$. For convenience we will simply refer