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SPINORS AND SPACE-TIME

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VOLUME 2
SPINOR AND TWISTOR METHODS
IN SPACE-TIME GEOMETRY

R.PENROSE & W.RINDLER

旋量与时空

第2卷

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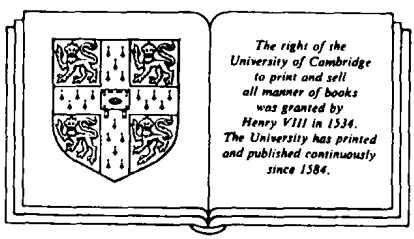
Volume 2
Spinor and twistor methods in space–time geometry

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Preface

This is a companion volume to our introductory work *Spinors and space–time*, Volume 1: *two-spinor calculus and relativistic fields*. There we attempted to demonstrate something of the power, utility and elegance of 2-spinor techniques in the study of space–time structure and physical fields, and to advocate the viewpoint that spinors may lie closer to the heart of (even macroscopic) physical laws than the vectors and tensors of the standard formalism. Here we carry these ideas further and discuss some important new areas of application. We introduce the theory of twistors and show how it sheds light on a number of important physical questions, one of the most noteworthy being the structure of energy–momentum/angular momentum of gravitating systems. The illumination that twistor theory brings to the discussion of such physical problems should lend further support to the viewpoint of an underlying spinorial structure in basic physical laws.

Those who have some familiarity with the standard 2-spinor formalism should be able to read this volume as an independent work. All necessary background material has been collected together in an introductory section which summarizes the relevant contents of Volume 1. There are many references to this earlier volume in the present work for the results and notations that are needed, but these are all explicitly provided in the summary, the numbering of results being the same as that of Volume 1 so that they can be located in either place without ambiguity. Detailed derivations are not repeated, however, and the reader who wishes to pursue these results *per se* would be well advised to refer back to the earlier volume.

The main topics introduced and discussed in this volume are twistor theory and related matters such as massless fields and the geometry of light rays, energy–momentum and angular momentum (from an unusual but, we hope, illuminating point of view), and the conformal structure of infinity. In addition we provide, in Chapter 8, a detailed classification of space–time curvature by 2-spinor means. Though we are mainly concerned with the classification of the Weyl (or empty-space Riemann) tensor, we also give a comprehensive treatment of the Ricci tensor, showing how this

arises as a particular case of the classification of symmetric spinors generally. Chapter 8 can be read independently of the other chapters in this volume, though there is some interrelation between its results and those of the other chapters.

Some remarks about the role of twistors in this volume are in order. Twistor theory can be viewed in two ways. On the one hand it may be seen simply as providing new mathematical techniques for the solution of problems within standard physical theory. On the other, it may be viewed as suggesting an alternative framework for the basis of all physics, characterized by the relegation of the concept of event (space–time point) from a primary to a secondary role. In this volume, twistors are treated very much in the former rather than in the latter way. This is not a work on twistor physics as such, though it can serve as a fairly comprehensive introduction to the subject for those who are interested in following up some of its more detailed, sophisticated, or speculative ideas.

We are concerned here mainly with the way that the mathematics of twistor theory interrelates with that of 2-component spinors. Spinor-field descriptions of twistors are presented in more detail than have been given hitherto. Many applications of twistor methods are given, such as: a proof that the vanishing of the Weyl tensor implies that the space–time is locally Minkowskian; the Kerr theorem for generating shear-free congruences in Minkowski space (including a generalization that applies in curved space–times); massless free-field contour integrals with basic sheaf cohomology ideas; the Ward construction for general self-dual Yang–Mills fields (though, to our regret, the original gravitational analogue of this construction is beyond the scope of this work – for we have felt it necessary to limit our discussion essentially to results with direct application to real, i.e. Lorentzian, space–times); conformal Killing vectors; Killing spinors; cosmological models; and the Grgin phenomenon for massless fields. An elaborate 'twistorial' discussion of energy–momentum and angular momentum in linearized gravitation is given, and this is generalized in our presentation of some recent developments which use corresponding twistor-type techniques in full general relativity and which provide a suggestive 'quasi-local' definition of mass. This leads us to a discussion of energy–momentum and angular momentum at null infinity and their behaviour under the Bondi–Metzner–Sachs (BMS) group – the group describing the asymptotic symmetries of an asymptotically flat space–time. We also give details of the full Bondi–Sachs mass-loss formula and present a version of Witten's argument for demonstrating mass positivity in general relativity as it applies at null infinity. None of this material on energy–

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momentum/angular momentum has appeared before in book form and it illustrates a significant new area in which spinor (and twistor) arguments can find powerful physical application. Much of the detailed spinor discussion of conformal infinity is presented here for the first time.

In an appendix we show how 2-spinors, Dirac 4-spinors, twistors and spin-weighted functions fit in with the general scheme of n-dimensional spinors.

As with Volume 1, there are many to whom we owe thanks, for assistance of a varied nature, either direct or indirect. Most particularly we are grateful to Michael Atiyah, Toby Bailey, Nick Buchdahl, Judith Daniels, Mike Eastwood, Robert Geroch, Denny Hill, Lane Hughston, Ben Jeffryes, Ron Kelly, Roy Kerr, Ted Newman, Zoltán Perjés, Ivor Robinson, Niall Ross, Ray Sachs, Engelbert Schücking, William Shaw, Iz Singer, Paul Sommers, George Sparling, Paul Tod, Helmuth Urbantke, Ronnie Wells and Nick Woodhouse. Again, our special thanks goes to Dennis Sciama for his encouragement and faith in this project for well over twenty years, and also to Tsou Sheung Tsun who has again provided invaluable help with the references.

Roger Penrose
Wolfgang Rindler

1985

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Summary of Volume 1

We provide here a connected account summarizing all the material of Volume 1 (*Two-spinor calculus and relativistic fields*) needed for the present volume. The equation numbering (and occasional section numbering) is precisely that of Volume 1, though sometimes the equations appear a little out of strict numerical order.

The light cone in Minkowski space

We use standard (*restricted*, i.e. right-handed and isochronous) Minkowski coordinates

$$x^0 = T, \quad x^1 = X, \quad x^2 = Y, \quad x^3 = Z$$

for (affine) Minkowski space $\mathbb{M}$ [*or* Minkowski vector space $\mathbb{V}$]. Points P, Q with coordinates $P^a = (P^0, P^1, P^2, P^3)$, $Q^a = (Q^0, Q^1, Q^2, Q^3)$, in $\mathbb{M}$, have invariant (squared) separation

$$\Phi(P,Q) = (Q^0 - P^0)^2 - (Q^1 - P^1)^2 - (Q^2 - P^2)^2 - (Q^3 - P^3)^2 \tag{1.1.22}$$

The *light cone* of P [*or* correspondingly the *null cone* in $\mathbb{V}$ when P is the origin] is the set of $Q \in \mathbb{M}$ [*or* $Q \in \mathbb{V}$] for which $\Phi(P,Q) = 0$. The light rays through P are the straight lines (generators) of the cone. The light rays through (say) the origin O can be parametrized by $\zeta \in \mathbb{C} \cup \{\infty\}$, where

$$\zeta = \frac{X + \mathrm{i}Y}{T - Z} = \frac{T + Z}{X - \mathrm{i}Y} \tag{1.2.19}$$

and, in terms of standard spherical polar coordinates on the sphere S^+, whose equation is

$$T = 1, \quad X^2 + Y^2 + Z^2 = 1,$$

we have

$$\zeta = \mathrm{e}^{\mathrm{i}\phi} \cot \frac{\theta}{2}. \tag{1.2.10}$$

S^+ is sometimes referred to as the *anti-celestial sphere*. The light rays

through O put it in one-to-one correspondence with the sphere S^- (given by $T = -1$, $X^2 + Y^2 + Z^2 = 1$): the *celestial sphere* of the observer at O with time-axis $X = Y = Z = 0$. The two are related by an antipodal map.

Under a (restricted) Lorentz transformation ζ transforms as

$$\zeta \to \tilde{\zeta} = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta} \tag{1.2.17}$$

where the coefficients α, β, γ, δ constitute the *spin-matrix*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad (\alpha\delta - \beta\gamma = 1; \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}).$$

This provides a (general) *conformal* motion of S^+, where we think of the points of S^+ as simply labelling the generators of the future light cone of P. The matrix acts on $\binom{\xi}{\eta}$, the components $\kappa^0 = \xi$, $\kappa^1 = \eta$ of a *spin-vector* $\boldsymbol{\kappa}$, where

$$\zeta = \xi/\eta = \kappa^0/\kappa^1.$$

The null vector ('flagpole') corresponding to this spin-vector has coordinates

$$T = \frac{1}{\sqrt{2}}(\xi\bar{\xi} + \eta\bar{\eta}), \qquad X = \frac{1}{\sqrt{2}}(\xi\bar{\eta} + \eta\bar{\xi}),$$

$$Y = \frac{1}{\mathrm{i}\sqrt{2}}(\xi\bar{\eta} - \eta\bar{\xi}), \qquad Z = \frac{1}{\sqrt{2}}(\xi\bar{\xi} - \eta\bar{\eta}), \tag{1.2.15}$$

(*cf.* also (3.1.31), (3.2.2)). *Unitary* spin-matrices describe rotations, while an active *boost* in the Z-direction of velocity v is described by

$$\begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} = \pm \begin{pmatrix} w^{\frac{1}{2}} & 0 \\ 0 & w^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{1.2.37}$$

with

$$w = \left(\frac{1+v}{1-v}\right)^{\frac{1}{2}}. \tag{1.2.36}$$

The Lorentz invariant cross-ratio of four null directions (or points of S^+) with complex parameters $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ is

$$\chi = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\} = \frac{(\zeta_1 - \zeta_2)(\zeta_3 - \zeta_4)}{(\zeta_1 - \zeta_4)(\zeta_3 - \zeta_2)}, \tag{1.3.9}$$

i.e.

$$\chi = \frac{(\xi_1\eta_2 - \xi_2\eta_1)(\xi_3\eta_4 - \xi_4\eta_3)}{(\xi_1\eta_4 - \xi_4\eta_1)(\xi_3\eta_2 - \xi_2\eta_3)}, \tag{1.3.10}$$

where $\zeta_j = \xi_j/\eta_j$. If the null directions are permuted, χ gets replaced by one

of

$$\chi, \quad 1-\chi, \quad \frac{1}{\chi}, \quad \frac{1}{1-\chi}, \quad \frac{\chi-1}{\chi}, \quad \frac{\chi}{\chi-1}. \tag{1.3.12}$$

The condition for the four null directions to lie in a hyperplane, i.e. for the corresponding points of S^+ to be concyclic, is that χ be real. The *harmonic* case $\chi = -1$ (or 2 or $\frac{1}{2}$) occurs when the four points can be transformed to the vertices of a square; and the *equianharmonic* case $\chi = -e^{\pm 2i\pi/3}$, to the vertices of a regular tetrahedron.

§1.5 Spinorial objects and spin structure

A non-zero spin-vector κ may be represented, up to a sign, as a null flag (i.e. a future-null vector with an oriented null 2-plane through it, cf. (3.2.2), (3.2.9); compare also Payne 1952) or as a pair of infinitesimally separated future-null directions. However, the *sign* of κ cannot be given a local geometrical interpretation since a continuous rotation through 2π (about any axis) sends κ into its negative, whereas it must restore any local geometrical structure (in the ordinary sense) to its original state. Thus we widen the concept of 'local geometry' to include *spinorial objects*, like κ, which are restored to their original states when rotated continuously through 4π but not through 2π. For spinorial objects to exist on a manifold $\mathcal{M}$, a restriction on its topology is needed, and then $\mathcal{M}$ is said to have *spin structure*. For spin-vectors to be defined consistently on a space–time manifold $\mathcal{M}$, we need to consider continuous motions of a null flag in which the point P at which it is situated is allowed to move continuously as well as rotations of the flag at P being allowed. Such a motion is called a *flag path*. For consistency $\mathcal{M}$ must, in addition to having spin structure, be space- and time-orientable – in which case $\mathcal{M}$ is said to have *spinor structure*. A (non-zero) spin-vector at point $P \in \mathcal{M}$ is an element of the (universal) double cover of the space of null flags at P; the (non-zero) spin-vector bundle of $\mathcal{M}$ is a (not necessarily universal) double cover of the null-flag bundle of $\mathcal{M}$. If $\mathcal{M}$ is not simply-connected it may possess inequivalent possible spin-vector bundles, i.e. different spinor structures. One two-fold ambiguity arises from each independent loop in $\mathcal{M}$ no odd multiple of which is continuously deformable to a point.

§2.2 The abstract-index formalism for tensor algebra

Tensor (or spinor) indices in *standard lightface italic type* are simply *abstract markers* and do not take numerical values or denote components in some

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basis frame. For example, the symbol V^a denotes the actual vector V paired with the marker a (an element of some pre-assigned labelling set $\mathscr{L}$); and V^b, V^{a_0}, etc., though representing the same geometrical vector V, are distinct elements of the abstract-index algebra because the vector is, in each case, paired with a different element, b, a_0, etc. of $\mathscr{L}$. Similarly, the spin-vector κ may be represented by various distinct elements κ^A, κ^{B_3}, etc., of the algebra, or by their complex conjugates $\bar{\kappa}^{A'}$, $\bar{\kappa}^{B'_3}$, etc. This notation allows commutativity (e.g. $U^a V^b = V^b U^a$, but $\neq V^a U^b$) in the expression for tensor product. Indices in *bold upright* type* are *numerical* and label *components* with respect to some chosen basis (e.g., world-tensor component indices **a**, **b**,... ranging over 0, 1, 2, 3 and spinor component indices **A**, **B**,... or **A'**, **B'**,... ranging over 0, 1 or 0', 1', respectively).

Denoting a *basis* (in the n-dimensional case) by $\delta^{\mathbf{a}}_{\mathbf{a}} = \delta^{\mathbf{a}}_1, \ldots \delta^{\mathbf{a}}_n$, with dual basis $\delta^{\mathbf{a}}_{\mathbf{a}} = \delta^1_{\mathbf{a}}, \ldots, \delta^n_{\mathbf{a}}$, we can express the *components* of a tensor $A^{\mathbf{a}\cdots\mathbf{\gamma}}_{\mathbf{\lambda}\cdots\mathbf{\nu}}$ as

$$A^{\mathbf{a}\cdots\mathbf{\gamma}}_{\mathbf{\lambda}\cdots\mathbf{\nu}} = A^{\mathbf{a}\cdots\mathbf{\gamma}}_{\mathbf{\lambda}\cdots\mathbf{\nu}} \delta^{\mathbf{a}}_{\mathbf{a}} \cdots \delta^{\mathbf{\gamma}}_{\mathbf{\gamma}} \delta^{\mathbf{\lambda}}_{\mathbf{\lambda}} \cdots \delta^{\mathbf{\nu}}_{\mathbf{\nu}} \tag{2.3.13}$$

where, inversely, the tensor itself can expressed in terms of its components by

$$A^{\mathbf{a}\cdots\mathbf{\gamma}}_{\mathbf{\lambda}\cdots\mathbf{\nu}} = A^{\mathbf{a}\cdots\mathbf{\gamma}}_{\mathbf{\lambda}\cdots\mathbf{\nu}} \delta^{\mathbf{a}}_{\mathbf{a}} \cdots \delta^{\mathbf{\gamma}}_{\mathbf{\gamma}} \delta^{\mathbf{\lambda}}_{\mathbf{\lambda}} \cdots \delta^{\mathbf{\nu}}_{\mathbf{\nu}}. \tag{2.3.14}$$

Different abstract index letters (which may be a mixture of upper and lower ones, if need be) may be 'clumped together' in the form of a single *composite* (abstract) *index* denoted, in general situations, by a capital script letter. Composite indices are useful for general statements about tensor (spinor) systems. The symbol $\mathfrak{S}^{\mathscr{A}}$ (or $\mathfrak{S}^{\mathscr{AB}}_{\mathscr{PQ}_0}$, for example, when the constituents of $\mathscr{A}$ are written explicitly) denotes the system ($\mathfrak{S}$-module) of tensors (spinors) whose abstract-index structure is that denoted by $\mathscr{A}$ in the upper position (the constituent indices being in either position). The scalars constitute a commutative ring $\mathfrak{S}$ with identity (normally C^∞ complex scalar fields on the space–time manifold $\mathscr{M}$). For real fields we use $\mathfrak{T}$ in place of $\mathfrak{S}$. The dual of $\mathfrak{S}^{\mathscr{A}}$ is $\mathfrak{S}_{\mathscr{A}}$ and, more generally, we have (for a totally reflexive system)

(2.2.38) PROPOSITION

The set of all $\mathfrak{S}$-multilinear maps from $\mathfrak{S}^{\mathscr{A}} \times \mathfrak{S}^{\mathscr{B}} \times \cdots \times \mathfrak{S}^{\mathscr{D}}$ to $\mathfrak{S}^{\mathscr{X}}$ may be identified with $\mathfrak{S}^{\mathscr{X}}_{\mathscr{AB}\ldots\mathscr{D}}$, where the maps are achieved by means of contracted product.

* In handwritten calculations it is convenient to distinguish such indices by underlining them with a wavy line.

Accordingly, any element $T^{\mathscr{X}}_{\mathscr{A}\mathscr{B}\ldots\mathscr{D}} \in \mathfrak{S}^{\mathscr{X}}_{\mathscr{A}\mathscr{B}\ldots\mathscr{D}}$ may be regarded as providing a map

$$T^{\mathscr{X}}_{\mathscr{A}\mathscr{B}\ldots\mathscr{D}} : \mathfrak{S}^{\mathscr{A}} \times \mathfrak{S}^{\mathscr{B}} \times \cdots \times \mathfrak{S}^{\mathscr{D}} \to \mathfrak{S}^{\mathscr{X}}$$

in addition to all the various other corresponding maps obtained by grouping the indices in different ways. Distinct script index letters generally indicate distinct clumpings of constituent tensor (or spinor) indices. If similar clumpings are required then suffixes are used on the composite indices, e.g. $\mathscr{A}_1, \mathscr{A}_2, \ldots$.

For spinors or tensors at a point P of a manifold $\mathscr{M}$ we may write, explicitly, $\mathfrak{S}^{\mathscr{A}}[P]$, and correspondingly $\mathfrak{S}^{\mathscr{A}}[\mathscr{U}]$ for the restrictions of the elements of $\mathfrak{S}^{\mathscr{A}}$ to a subset $\mathscr{U}$ of $\mathscr{M}$.

Tensor–spinor abstract index correspondence

World-tensor indices are regarded *as* composite indices, each simply standing for a clumped pair of corresponding spinor indices, one primed and one unprimed. We adopt the standard convention

$$a = AA', \quad b = BB', \quad c = CC', \quad \ldots, \quad z = ZZ',$$
$$a_0 = A_0A_0', \quad \ldots, \quad a_1 = A_1A_1', \quad \ldots \tag{3.1.2}$$

Primed spinor indices are the complex conjugates of their corresponding unprimed ones (in the sense $\overline{\phi^{AB'}} = \bar{\phi}^{A'B}$ etc.). Spinor indices are raised and lowered with the anti-symmetrical ε-spinors:

$$\psi^{\mathscr{C}A} = \varepsilon^{AB}\psi^{\mathscr{C}}{}_B = -\psi^{\mathscr{C}}{}_B\varepsilon^{BA} \tag{2.5.14}$$

$$\psi^{\mathscr{C}}{}_B = \psi^{\mathscr{C}A}\varepsilon_{AB} = -\varepsilon_{BA}\psi^{\mathscr{C}A} \tag{2.5.15}$$

(and similarly for primed indices, with $\varepsilon^{A'B'} := \bar{\varepsilon}^{A'B'} = \overline{\varepsilon^{AB}}$, $\varepsilon_{A'B'} := \bar{\varepsilon}_{A'B'} = \overline{\varepsilon_{AB}}$), this being consistent with the normal world-tensor index raising and lowering (using g_{ab} and g^{ab}), where

$$g_{ab} = \varepsilon_{AB}\varepsilon_{A'B'}, \quad g_a{}^b = \varepsilon_A{}^B\varepsilon_{A'}{}^{B'}, \quad g^{ab} = \varepsilon^{AB}\varepsilon^{A'B'}. \tag{3.1.9}$$

(Note that $g_a{}^b = g^b{}_a$, $\varepsilon_A{}^B = -\varepsilon^B{}_A$ and $\varepsilon_{A'}{}^{B'} = -\varepsilon^{B'}{}_{A'}$ are 'Kronecker delta' symbols.) While we must be careful to preserve the ordering of the unprimed indices on a spinor symbol, and also of the primed ones, the *relative* ordering between primed and unprimed indices is immaterial; for example $\psi^a{}_b{}^Q$ is each of the following except the last:

$$\psi^{AA'}{}_{B'B}{}^Q = \psi^{A'A}{}_B{}^Q{}_{B'} = \psi^A{}_B{}^{QA'}{}_{B'} = \psi^{AA'Q}_{BB'} \neq \psi^{AA'Q}{}_{BB'}. \tag{2.5.33}$$

The ε-spinors satisfy the identities

$$\varepsilon_{AB}\varepsilon_{CD} + \varepsilon_{BC}\varepsilon_{AD} + \varepsilon_{CA}\varepsilon_{BD} = 0, \tag{2.5.21}$$

their raised versions such as

$$\varepsilon_{AB}\varepsilon_C{}^D + \varepsilon_{BC}\varepsilon_A{}^D + \varepsilon_{CA}\varepsilon_B{}^D = 0, \tag{2.5.20}$$

$$\varepsilon_A{}^C\varepsilon_B{}^D - \varepsilon_B{}^C\varepsilon_A{}^D = \varepsilon_{AB}\varepsilon^{CD}, \tag{2.5.22}$$

and also the complex conjugates of these relations. We deduce

$$\phi_{\mathcal{Q}AB} - \phi_{\mathcal{Q}BA} = \phi_{\mathcal{Q}C}{}^C\varepsilon_{AB}, \tag{2.5.23}$$

so for $\phi_{\mathcal{Q}AB}$ *skew* in A, B,

$$\phi_{\mathcal{Q}AB} = \tfrac{1}{2}\phi_{\mathcal{Q}C}{}^C\varepsilon_{AB}. \tag{2.5.24}$$

Bases and components

The standard symbols for a *spinor basis* for $\mathcal{S}^A$, or *dyad*, are o^A, ι^A, where $o^{A'} := \bar{o}^{A'}(= \overline{o^A})$ and $\iota^{A'} := \bar{\iota}^{A'}(= \overline{\iota^A})$. These need not be normalized to unity, and we define

$$\chi = \varepsilon_{AB}o^A\iota^B = o_A\iota^A. \tag{2.5.46}$$

When $\chi = 1$, the dyad is a *spin-frame* and we have

$$\varepsilon^{AB} = o^A\iota^B - \iota^A o^B, \quad \varepsilon_{AB} = o_A\iota_B - \iota_A o_B, \quad \varepsilon_A{}^B = o_A\iota^B - \iota_A o^B. \tag{2.5.54}$$

The condition for a general dyad, on the other hand, is simply $\chi \neq 0$, since

(2.5.56) PROPOSITION

The condition $\alpha_A\beta^A = 0$ at a point is necessary and sufficient for α_A, β_A to be scalar multiples of each other at that point.

Associated with any spin-frame is a *null tetrad* l^a, n^a, m^a, $\bar{m}^a$ defined by

$$\begin{aligned} l^a &= o^A\bar{o}^{A'}, \quad & n^a &= \iota^A\bar{\iota}^{A'}, \\ m^a &= o^A\bar{\iota}^{A'}, \quad & \bar{m}^a &= \iota^A\bar{o}^{A'}, \end{aligned} \tag{3.1.14}$$

and satisfying

$$l^a l_a = n^a n_a = m^a m_a = \bar{m}^a\bar{m}_a = 0, \tag{3.1.15}$$

$$l^a n_a = 1, \quad m^a\bar{m}_a = -1, \tag{3.1.16}$$

$$l^a m_a = l^a\bar{m}_a = n^a m_a = n^a\bar{m}_a = 0, \tag{3.1.17}$$

$$l^a = \bar{l}^a, \quad n^a = \bar{n}^a, \tag{3.1.18}$$

and also a standard restricted Minkowski tetrad

$$t^a = \frac{1}{\sqrt{2}}(l^a + n^a) = \frac{1}{\sqrt{2}}(o^A\bar{o}^{A'} + \iota^A\bar{\iota}^{A'}),$$

$$x^a = \frac{1}{\sqrt{2}}(m^a + \bar{m}^a) = \frac{1}{\sqrt{2}}(o^A \iota^{A'} + \iota^A o^{A'}),$$

$$y^a = \frac{\mathrm{i}}{\sqrt{2}}(m^a - \bar{m}^a) = \frac{\mathrm{i}}{\sqrt{2}}(o^A \iota^{A'} - \iota^A o^{A'}),$$

$$z^a = \frac{1}{\sqrt{2}}(l^a - n^a) = \frac{1}{\sqrt{2}}(o^A o^{A'} - \iota^A \iota^{A'}), \tag{3.1.20}$$

where reciprocally

$$l^a = \frac{1}{\sqrt{2}}(t^a + z^a) = o^A o^{A'},$$

$$n^a = \frac{1}{\sqrt{2}}(t^a - z^a) = \iota^A \iota^{A'},$$

$$m^a = \frac{1}{\sqrt{2}}(x^a - \mathrm{i}y^a) = o^A \iota^{A'},$$

$$\bar{m}^a = \frac{1}{\sqrt{2}}(x^a + \mathrm{i}y^a) = \iota^A o^{A'}. \tag{3.1.21}$$

See Fig. 1–17 for the geometrical relation between o^A, ι^A and t^a, x^a, y^a, z^a. The components of a vector K^a in the Minkowski tetrad are related to the

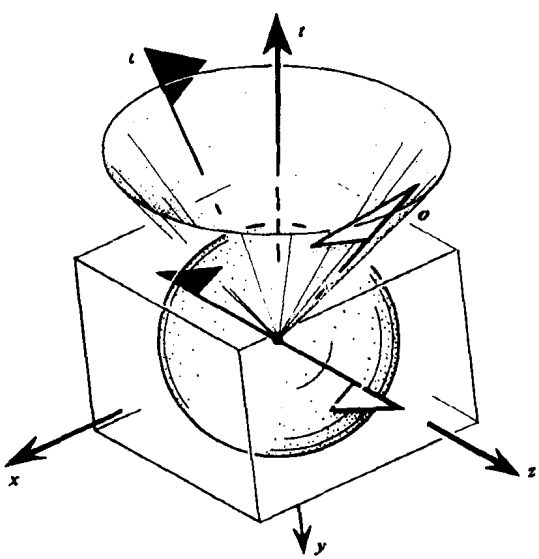


Fig. 1-17. The standard relation between a spin-frame o, ι and a (restricted) Minkowski tetrad t, x, y, z.

spin-frame components of $K^{AA'}$ by

$$\frac{1}{\sqrt{2}}\begin{pmatrix} K^0+K^3 & K^1+iK^2 \\ K^1-iK^2 & K^0-K^3 \end{pmatrix}=\begin{pmatrix} K^{00'} & K^{01'} \\ K^{10'} & K^{11'} \end{pmatrix}. \tag{3.1.31}$$

To express the corresponding relations between Minkowski tetrad components and spin-frame components for a general tensor it is often convenient to use the translation symbols:

$$g_0{}^{AB'}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&1\end{pmatrix}=g_{AB'}{}^0,\quad g_1{}^{AB'}=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\1&0\end{pmatrix}=g_{AB'}{}^1,$$

$$g_2{}^{AB'}=\frac{1}{\sqrt{2}}\begin{pmatrix}0&i\\-i&0\end{pmatrix}=-g_{AB'}{}^2,\quad g_3{}^{AB'}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=g_{AB'}{}^3. \tag{3.1.49}$$

Null vectors and null flags

A complex null vector χ^a, in the normal sense $\chi^a\chi_a=0$, has the spinor form

$$\chi^a=\kappa^A\xi^{A'} \tag{3.2.6}$$

where we can take $\xi^{A'}=\bar{\kappa}^{A'}$ [*or* $-\bar{\kappa}^{A'}$] whenever χ^a is real and future-null [past-null] (or zero). Two complex null vectors χ^a and ψ^a, with $\chi^a\neq 0$ given by (3.2.6), are orthogonal in the sense

$$\chi_a\psi^a=0 \tag{3.2.22}$$

if and only if $\psi^a=\kappa^A\eta^{A'}$ or $\tau^A\xi^{A'}$ for some $\eta^{A'}$ or τ^A.

In addition to defining the real future-null vector

$$K^a=\kappa^A\bar{\kappa}^{A'}, \tag{3.2.2}$$

called its *flagpole*, any non-zero spin-vector κ^A determines the real simple null bivector

$$P^{ab}=\kappa^A\kappa^B\varepsilon^{A'B'}+\varepsilon^{AB}\bar{\kappa}^{A'}\bar{\kappa}^{B'} \tag{3.2.9}$$

which defines an oriented null 2-plane element (through the flagpole) called the *flag plane* (or an 'axe' by Payne 1952). The flagpole and flag plane together define the *null flag* which represents κ^A up to sign. Multiplying κ^A by $re^{i\theta}$ (r,θ real) increases the flagpole's extent by r^2 and rotates the flag plane by 2θ.

§3.3 Symmetry operations

We adopt the standard convention that round brackets (parentheses) around indices denote that the symmetric part is being taken and square

brackets, the antisymmetric part. Vertical bars are used at either side of a set of indices to be omitted from an (anti-) symmetrization. If a pair of brackets appears nested within another, then the inside operation is to be performed first and the resulting indices (if not blocked off by vertical bars) then partake again of the larger (anti-) symmetry operation. (Though such a notation is, strictly speaking, redundant it can be very useful in the formulation of a proof, e.g. see (4.3.17).) While the bracket notation normally applies just to individual tensors or spinors, it may also be used to denote *subspaces* of $\mathfrak{S}_{\mathscr{A}}$ for which (anti-) symmetry relations are imposed. Thus, for example:

$$\chi_{\mathscr{A}}{}^{\alpha}{}_{\lambda\mu}{}^{\beta}{}_{\nu}{}^{\mathscr{B}_1\mathscr{B}_2}\in\mathfrak{S}^{[\alpha\beta](\mathscr{B}_1\mathscr{B}_2)}_{\mathscr{A}(\lambda\mu\nu)} \quad \textit{iff} \quad \chi_{\mathscr{A}}{}^{\alpha}{}_{\lambda\mu}{}^{\beta}{}_{\nu}{}^{\mathscr{B}_1\mathscr{B}_2}=\chi_{\mathscr{A}}{}^{[\alpha}{}_{(\lambda\mu}{}^{\beta]}{}_{\nu)}{}^{(\mathscr{B}_1\mathscr{B}_2)}. \tag{3.3.14}$$

Various simple results concerning symmetries are useful. For example (with Greek letters denoting n-dimensional abstract tensor indices):

(3.2.22) PROPOSITION

$$\phi_{\mathscr{A}\alpha\beta\ldots\delta}X^{\alpha}X^{\beta}\ldots X^{\delta}=0 \textit{ for all } X^{\alpha}\in\mathfrak{S}^{\alpha} \textit{ iff } \phi_{\mathscr{A}(\alpha\ldots\delta)}=0.$$

Hence

the function $\phi_{\mathscr{A}}(X)\equiv\phi_{\mathscr{A}\alpha\ldots\delta}X^{\alpha}\ldots X^{\delta}$ *serves to define the tensor* $\phi_{\mathscr{A}(\alpha\ldots\delta)}$ *uniquely.* (3.3.23)

Combinations of symmetries and anti-symmetries can be applied to produce a tensor with a *Young tableau* symmetry and which is, accordingly, (pointwise) *irreducible* under the group of general linear transformations (about that point). Some relevant properties are discussed in the extended 'footnote' on p. 143 of Volume 1. In particular, a tensor $\phi_{\alpha\ldots\lambda}$ with tableau symmetry – of the version in which groups of non-increasing lengths of *symmetric* (rather than anti-symmetric) indices are exhibited – may be represented by a polynomial $P(X^{\alpha},\ldots,Z^{\alpha})$, homogeneous of non-increasing degrees in $X^{\alpha},\ldots,Z^{\alpha}$ separately. (Here (3.3.23) is being applied to *each* symmetric group of indices of $\phi_{\ldots\ldots}$.) The *tableau property* (i.e. 'hidden saturation by anti-symmetries') may then be expressed as:

$$X^{\alpha}\partial P/\partial Y^{\alpha}=0,\ldots,X^{\alpha}\partial P/\partial Z^{\alpha}=0,\ldots,Y^{\alpha}\partial P/\partial Z^{\alpha}=0$$

(earlier variable contracted with derivative with respect to later variable) in which case P can be expressed as a function just of the skew products like $X^{[\alpha}Y^{\beta}Z^{\gamma]}$, $X^{[\alpha}Y^{\beta]}$, X^{α} (consecutive variables starting with X^{α}). Equivalently, the tableau property may be expressed as *any further symmetrization of* $\phi_{\alpha\ldots\lambda}$ *which includes one entire symmetric group of indices together with one index from a later group, vanishes.* For such a $\phi_{\alpha\ldots\lambda}$, if

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