

Dynamic Modeling and Econometrics in
Economics and Finance 21

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Derivative Security Pricing

Techniques, Methods and Applications



Springer

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Dynamic Modeling and Econometrics in Economics and Finance

Volume 21

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Preface

This book is an outgrowth of courses we have offered on stochastic calculus and its applications to derivative securities pricing over the last 15 years. The courses have been offered several times to doctoral students and students in the Master of Quantitative Finance and its forerunner programs in the Finance Discipline Group, UTS Business School at the University of Technology Sydney (UTS), and three times to students in the Financial Engineering program at Nanyang Business School in Singapore. It has also served for shorter courses at the Graduate School of International Corporate Strategy at Hitotsubashi University in Tokyo, the Faculty of Economics at the University of Bielefeld in Germany, the Dipartimento di Matematica per le Decisioni at the University of Florence and the Graduate School of Economics at the University of Kyoto.

The aim of the book is to provide a unifying framework within which many of the key results on derivative security pricing can be placed and related to each other. We have also tried to provide an introductory discussion on stochastic processes sufficient to give a good intuitive feel for Ito's Lemma, martingales and the application of Girsanov's theorem. With the explosion of the literature on option pricing in the last four decades, it would obviously not have been possible to cover in one course even a fraction of the main results. Rather, it was our intention that those completing the course would be able to more confidently approach that literature with a good intuitive understanding of the basic techniques, a good overview of how the different parts of the literature relate to each other, and a knowledge of how to implement the theory for their own particular problems. Judging from the feedback we have received, the book has been successful in these aims, and we have been heartened by the very positive response we have had from people who have read it. This includes not only our immediate circle of research collaborators and doctoral students at UTS but also students, researchers and practitioners both in Australia and overseas. The feedback we have received has left us more convinced than we were 15 years ago that this book fills an important gap in the pedagogical finance literature. There are now many excellent monographs and survey papers that treat the revolution of stochastic methods in finance over the last 40 years. However, many of these treatments require of the reader a high degree of, if not "fluency",

then certainly “maturity” with the concepts of measure theory or, are written in the very formal lemma/theorem/proof style of modern mathematics. Readers who are not comfortable with these concepts and formal mathematical approaches are left with a feeling of not understanding the essential foundations of the subject and always lack confidence in applying the techniques of stochastic finance. Our aim in this book is to present to the reader a treatment which emphasises more the financial intuition of the material, and which is at the same mathematical level and uses the same basic hedging arguments of the early papers of Black–Scholes and Merton, which sparked off the revolution to which we have referred above. Uppermost in our mind has been the desire to give the reader an intuitive feel for the many difficult mathematical concepts that will be encountered in working through this book. Whilst the mathematical level is demanding, it should nevertheless be attainable for readers who are comfortable with an intermediate level of calculus and the non-measure theoretic approach to probability theory.

By the foregoing remarks, we do not intend to downplay or denigrate the importance of the modern measure theoretic approach to the theory of diffusion processes and semimartingale integration. We are acutely aware of the fact that many of the subtleties of stochastic finance require these advanced techniques for their proper elucidation. Furthermore, many of the important advances of the last three decades would not have occurred, or would have been much slower in coming, without their use. However, stochastic finance is rapidly evolving from its pure science phase to its applied science and engineering phase. As a result, there is a greater influx into the area of academics and practitioners who are neither “fluent” nor even “comfortable” with measure theoretic arguments and the formal style of modern mathematics. It is to this audience that this book is addressed. Of course the challenge in writing a book at a more intuitive level is to do so in a way that is respectful of the many subtleties that the measure theoretic approach and more formal mathematical approach have been developed to address. We have done our best to meet this challenge; however, we are mindful of many shortcomings that may still exist.

Another feature of the book is the set of problems that has been developed to accompany each chapter. Here, we have tried to include exercises that cover many of the key results and examples that have become significant in applications or in subsequent theoretical developments. As we very firmly believe that a full understanding of stochastic methods in finance can only be attained when one can simulate and compute the quantities that one is discussing, we have also included a number of computational exercises.

The evolution of our thinking about stochastic methods in finance has been greatly assisted by John Van der Hoek of the University of South Australia and our UTS colleague Eckhard Platen. They have been most generous in sharing their knowledge both in private conversations and in courses which they have kindly presented at UTS. We would also like to thank some of our former doctoral students, in particular Nadima El-Hassan, Garry de Jager, Ramaprasad Bhar, Oh Kang Kwon, Adam Kucera, Shenhui Gao, Thuy Duong Tô and Andrew Ziogas. Numerous discussions and debates with them over recent years have helped, if not them,

then certainly ourselves to clarify a number of technical points discussed in this book. Thanks are also due to Andrew Ziogas, Nicole Mingxi Huang and Hing Hung for developing the MATLAB programs used to do the various simulations. We are grateful to Mark Craddock and Boda Kang for checking through a number of mathematical derivations and making some valuable suggestions. We are also indebted to Simon Carlstedt for checking thoroughly through the book and pointed out a number of errors and inconsistencies. Finally, we would like to acknowledge the efforts of Xiaolin Miao, Yuping Wu, Jingfeng He, Xuli Huang, Lifang Zhang, Jenny Yixin Chen, Shing-Yih Chai, Laura Santuz, Gwen Tran, Stephanie Ji-Won Ough and Linh Thuy Tô who have worked diligently and under much pressure to prepare the various drafts and the many graphs. However, all of the aforementioned persons should be totally absolved from any blame for any errors, omissions or confusions that this book may still contain.

University of Technology Sydney
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Part I
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