

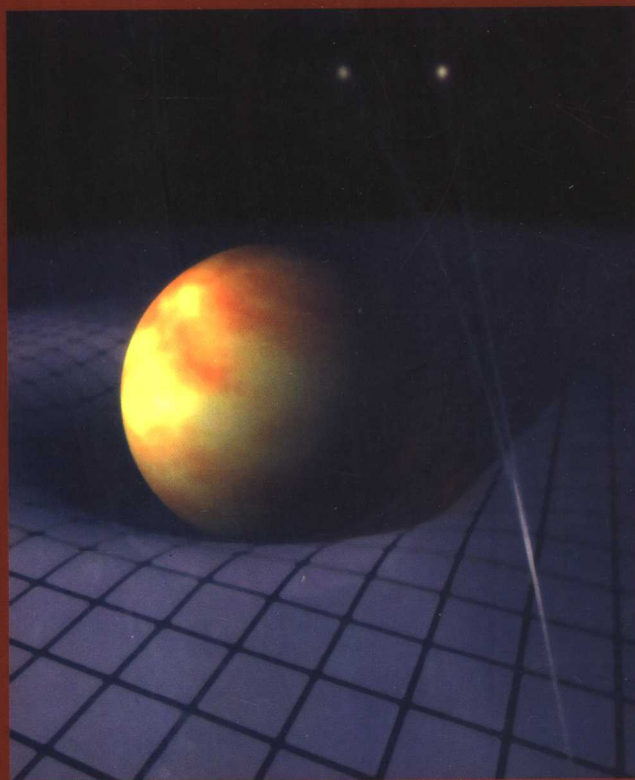
An Introduction to General Relativity

SPACETIME

and

GEOMETRY

时空与几何



Sean M. Carroll

世界图书出版公司

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图书在版编目 (C I P) 数据

时空与几何: 英文 / (美) 卡莱尔 (Carroll, S.) 著.
北京: 世界图书出版公司北京公司, 2007. 10
书名原文: Spacetime and Geometry
ISBN 978-7-5062-8342-7

I. 时… II. 卡… III. ①时空—研究—英文②几何—研
究—英文 IV. 0412.1 018

中国版本图书馆CIP数据核字 (2007) 第140723号

书 名: Spacetime and Geometry

作 者: Sean M. Carroll

中 译 名: 时空与几何

责任编辑: 高蓉 刘慧

出 版 者: 世界图书出版公司北京公司

印 刷 者: 三河市国英印务有限公司

发 行: 世界图书出版公司北京公司 (北京朝内大街 137 号 100010)

联系电话: 010-64015659, 64038348

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开 本: 16 开

印 张: 33.5

版 次: 2007 年 10 月第 1 次印刷

版权登记: 图字:01-2007-4306

书 号: 978-7-5062-8342-7 / O · 614

定 价: 69.00 元

世界图书出版公司北京公司已获得 Pearson Education 授权在中国大陆独家重印发行

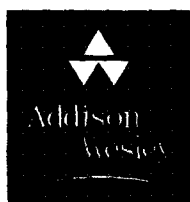
Acquisitions Editor: Adam Black
Project Editor: Nancy Benton
Text Designer: Leslie Galen
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Marketing Manager: Christy Lawrence
Manufacturing Coordinator: Vivian McDougal
Project Coordination and Electronic Page Makeup: Integre Technical Publishing Co., Inc.

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Carroll, Sean

ISBN 0-8053-8732-3 (hardcover)



5 6 7 8 9 10 —MAL— 06

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“For if each Star is little more a mathematical Point, located upon the Hemisphere of Heaven by Right Ascension and Declination, then all the Stars, taken together, tho’ innumerable, must like any other set of points, in turn represent some single gigantick Equation, to the mind of God as straightforward as, say, the Equation of a Sphere,—to us unreadable, incalculable. A lonely, uncompensated, perhaps even impossible Task,—yet some of us must ever be seeking, I suppose.”

—Thomas Pynchon, *Mason & Dixon*

Preface

General relativity is the most beautiful physical theory ever invented. It describes one of the most pervasive features of the world we experience—gravitation—in terms of an elegant mathematical structure—the differential geometry of curved spacetime—leading to unambiguous predictions that have received spectacular experimental confirmation. Consequences of general relativity, from the big bang to black holes, often get young people first interested in physics, and it is an unalloyed joy to finally reach the point in one's studies where these phenomena may be understood at a rigorous quantitative level. If you are contemplating reading this book, that point is here.

In recent decades, general relativity (GR) has become an integral and indispensable part of modern physics. For a long time after it was proposed by Einstein in 1916, GR was counted as a shining achievement that lay somewhat outside the mainstream of interesting research. Increasingly, however, contemporary students in a variety of specialties are finding it necessary to study Einstein's theory. In addition to being an active research area in its own right, GR is part of the standard syllabus for anyone interested in astrophysics, cosmology, string theory, and even particle physics. This is not to slight the more pragmatic uses of GR, including the workings of the Global Positioning System (GPS) satellite network.

There is no shortage of books on GR, and many of them are excellent. Indeed, approximately thirty years ago witnessed the appearance of no fewer than three books in the subject, each of which has become a classic in its own right: those by Weinberg (1972), Misner, Thorne, and Wheeler (1973), and Hawking and Ellis (1975). Each of these books is suffused with a strongly-held point of view advocated by the authors. This has led to a love-hate relationship between these works and their readers; in each case, it takes little effort to find students who will declare them to be the best textbook ever written, or other students who find them completely unpalatable. For the individuals in question, these judgments may very well be correct; there are many different ways to approach this subject.

The present book has a single purpose: to provide a clear introduction to general relativity, suitable for graduate students or advanced undergraduates. I have attempted to include enough material so that almost any one-semester introductory course on GR can find the appropriate subjects covered in the text, but not too much more than that. In particular, I have tried to resist the temptation to write a comprehensive reference book. The only goal of this book is to teach you GR.

An intentional effort has been made to prefer the conventional over the idiosyncratic. If I can be accused of any particular ideological bias, it would be a

tendency to think of general relativity as a field theory, a point of view that helps one to appreciate the connections among GR, particle physics, and string theory. At the same time, there are a number of exciting astrophysical applications of GR (black holes, gravitational lensing, the production and detection of gravitational waves, the early universe, the late universe, the cosmological constant), and I have endeavored to include at least enough background discussion of these issues to prepare students to tackle the current literature.

The primary question facing any introductory treatment of general relativity is the level of mathematical rigor at which to operate. There is no uniquely proper solution, as different students will respond with different levels of understanding and enthusiasm to different approaches. Recognizing this, I have tried to provide something for everyone. I have not shied away from detailed formalism, but have also attempted to include concrete examples and informal discussion of the concepts under consideration. Much of the most mathematical material has been relegated to the Appendices. Some of the material in the Appendices is actually an integral part of the course (for example, the discussion of conformal diagrams), but an individual reader or instructor can decide just when it is appropriate to delve into them; signposts are included in the body of the text.

Surprisingly, there are very few formal prerequisites for learning general relativity; most of the material is developed as we go along. Certainly no prior exposure to Riemannian geometry is assumed, nor would it necessarily be helpful. It would be nice to have already studied some special relativity; although a discussion is included in Chapter 1, its purpose is more to review the basics and to introduce some notation, rather than to provide a self-contained introduction. Beyond that, some exposure to electromagnetism, Lagrangian mechanics, and linear algebra might be useful, but the essentials are included here.

The structure of the book should be clear. The first chapter is a review of special relativity and basic tensor algebra, including a brief discussion of classical field theory. The next two chapters introduce manifolds and curvature in some detail; some motivational physics is included, but building a mathematical framework is the primary goal. General relativity proper is introduced in Chapter 4, along with some discussion of alternative theories. The next four chapters discuss the three major applications of GR: black holes (two chapters), perturbation theory and gravitational waves, and cosmology. Each of these subjects has witnessed an explosion of research in recent years, so the discussions here will be necessarily introductory, but I have tried to emphasize issues of relevance to current work. These three applications can be covered in any order, although there are interdependencies highlighted in the text. Discussions of experimental tests are sprinkled through these chapters. Chapter 9 is a brief introduction to quantum field theory in curved spacetime; this is not a necessary part of a first look at GR, but has become increasingly important to work in quantum gravity and cosmology, and therefore deserves some mention. On the other hand, a few topics are scandalously neglected; the initial-value problem and cosmological perturbation theory come to mind, but there are others. Fortunately there is no shortage of other resources. The Appendices serve various purposes: There are discussions of

technical points that were avoided in the body of the book, crucial concepts that could have been put in various places, and extra topics that are useful but outside the main development.

Since the goal of the book is pedagogy rather than originality, I have often leaned heavily on other books (listed in the bibliography) when their expositions seemed perfectly sensible to me. When this leaning was especially heavy, I have indicated it in the text itself. It will be clear that a primary resource was the book by Wald (1984), which has become a standard reference in the field; readers of this book will hopefully be well-prepared to jump into the more advanced sections of Wald's book.

This book grew out of a set of lecture notes that were prepared when I taught a course on GR at MIT. These notes are available on the web for free, and will continue to be so; they will be linked to the website listed below. Perhaps a little over half of the material here is contained in the notes, although the advantages of owning the book (several copies, even) should go without saying.

Countless people have contributed greatly both to my own understanding of general relativity and to this book in particular—too many to acknowledge with any hope of completeness. Some people, however, deserve special mention. Ted Pyne learned the subject along with me, taught me a great deal, and collaborated with me the first time we taught a GR course, as a seminar in the astronomy department at Harvard; parts of this book are based on our mutual notes. Nick Warner taught the course at MIT from which I first learned GR, and his lectures were certainly a very heavy influence on what appears here. Neil Cornish was kind enough to provide a wealth of exercises, many of which have been included at the end of each chapter. And among the many people who have read parts of the manuscript and offered suggestions, Sanaz Arkani-Hamed was kind enough to go through the entire thing in great detail.

I would also like to thank everyone who either commented in person or by email on different parts of the book; these include Tigran Aivazian, Teodora Beloshka, Ed Bertschinger, Patrick Brady, Peter Brown, Jennifer Chen, Michele Ferraz Figueiró, Eanna Flanagan, Jacques Fric, Ygor Geurts, Marco Godina, Monica Guica, Jim Hartle, Tamás Hauer, Daniel Holz, Ted Jacobson, Akash Kansagra, Chuck Keeton, Arthur Kosowsky, Eugene Lim, Jorma Louko, Robert A. McNees, Hayri Mutluay, Simon Ross, Itai Seggev, Robert Wald, and Barton Zwiebach. Apologies are due to anyone I may have neglected to mention. And along the way I was fortunate to be the recipient of wisdom and perspective from numerous people, including Shadi Bartsch, George Field, Deryn Fogg, Ilana Harus, Gretchen Helfrich, Mari Ruti, Maria Spiropulu, Mark Trodden, and of course my family. (This wisdom often came in the form, "What were you thinking?") Finally, I would like to thank the students in my GR classes, on whom the strategies deployed here were first tested, and express my gratitude to my students and collaborators, for excusing my book-related absences when I should have been doing research.

My friends who have written textbooks themselves tell me that the first printing of a book will sometimes contain mistakes. In the unlikely event that this happens

here, there will be a list of errata kept at the website for the book:

<http://spacetimeandgeometry.net/>

The website will also contain other relevant links of interest to readers.

During the time I was working on this book, I was supported by the National Science Foundation, the Department of Energy, the Alfred P. Sloan Foundation, and the David and Lucile Packard Foundation.

Sean Carroll
Chicago, Illinois
June 2003

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CHAPTER

1

Special Relativity and Flat Spacetime

1.1 ■ PRELUDE

General relativity (GR) is Einstein's theory of space, time, and gravitation. At heart it is a very simple subject (compared, for example, to anything involving quantum mechanics). The essential idea is perfectly straightforward: while most forces of nature are represented by fields defined on spacetime (such as the electromagnetic field, or the short-range fields characteristic of subnuclear forces), gravity is inherent in spacetime itself. In particular, what we experience as "gravity" is a manifestation of the *curvature* of spacetime.

Our task, then, is clear. We need to understand spacetime, we need to understand curvature, and we need to understand how curvature becomes gravity. Roughly, the first two chapters of this book are devoted to an exploration of spacetime, the third is about curvature, and the fourth explains the relationship between curvature and gravity, before we get into applications of the theory. However, let's indulge ourselves with a short preview of what is to come, which will perhaps motivate the initial steps of our journey.

GR is a theory of gravity, so we can begin by remembering our previous theory of gravity, that of Newton. There are two basic elements: an equation for the gravitational field as influenced by matter, and an equation for the response of matter to this field. The conventional Newtonian statement of these rules is in terms of forces between particles; the force between two objects of masses M and m separated by a vector $\mathbf{r} = r\mathbf{e}_{(r)}$ is the famous inverse-square law,

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e}_{(r)}, \quad (1.1)$$

and this force acts on a particle of mass m to give it an acceleration according to Newton's second law,

$$\mathbf{F} = m\mathbf{a}. \quad (1.2)$$

Equivalently, we could use the language of the gravitational potential Φ ; the potential is related to the mass density ρ by Poisson's equation,

$$\nabla^2\Phi = 4\pi G\rho, \quad (1.3)$$

and the acceleration is given by the gradient of the potential,

$$\mathbf{a} = -\nabla\Phi. \quad (1.4)$$

Either (1.1) and (1.2), or (1.3) and (1.4), serve to define Newtonian gravity. To define GR, we need to replace each of them by statements about the curvature of spacetime.

The hard part is the equation governing the response of spacetime curvature to the presence of matter and energy. We will eventually find what we want in the form of Einstein's equation,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}. \quad (1.5)$$

This looks more forbidding than it should, largely because of those Greek subscripts. In fact this is simply an equation between 4×4 matrices, and the subscripts label elements of each matrix. The expression on the left-hand side is a measure of the curvature of spacetime, while the right-hand side measures the energy and momentum of matter, so this equation relates energy to curvature, as promised. But we will defer until later a detailed understanding of the inner workings of Einstein's equation.

The response of matter to spacetime curvature is somewhat easier to grasp: Free particles move along paths of "shortest possible distance," or geodesics. In other words, particles try their best to move on straight lines, but in a curved spacetime there might not be any straight lines (in the sense we are familiar with from Euclidean geometry), so they do the next best thing. Their parameterized paths $x^\mu(\lambda)$ obey the geodesic equation:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0. \quad (1.6)$$

At this point you aren't expected to understand (1.6) any more than (1.5); but soon enough it will all make sense.

As we will discuss later, the universal nature of geodesic motion is an extremely profound feature of GR. This universality is the origin of our claim that gravity is not actually a "force," but a feature of spacetime. A charged particle in an electric field feels an acceleration, which deflects it from straight-line motion; in contrast, a particle in a gravitational field moves along a path that is the closest thing there is to a straight line. Such particles do not feel acceleration; they are freely falling. Once we become more familiar with the spirit of GR, it will make perfect sense to think of a ball flying through the air as being more truly "unaccelerated" than one sitting on a table; the one sitting on a table is being deflected away from the geodesic it would like to be on (which is why we feel a force on our feet as we stand on Earth).

The basic concept underlying our description of spacetime curvature will be that of the metric tensor, typically denoted by $g_{\mu\nu}$. The metric encodes the geometry of a space by expressing deviations from Pythagoras's theorem, $(\Delta l)^2 = (\Delta x)^2 + (\Delta y)^2$ (where Δl is the distance between two points defined on a Cartesian grid with coordinate separations Δx and Δy). This familiar formula is valid only in conventional Euclidean geometry, where it is implicitly assumed that space is flat. In the presence of curvature our deeply ingrained notions of ge-

ometry will begin to fail, and we can characterize the amount of curvature by keeping track of how Pythagoras's relation is altered. This information is contained in the metric tensor. From the metric we will derive the Riemann curvature tensor, used to define Einstein's equation, and also the geodesic equation. Setting up this mathematical apparatus is the subject of the next several chapters.

Despite the need to introduce a certain amount of formalism to discuss curvature in a quantitative way, the essential notion of GR ("gravity is the curvature of spacetime") is quite simple. So why does GR have, at least in some benighted circles, a reputation for difficulty or even abstruseness? Because the elegant truths of Einstein's theory are obscured by the accumulation of certain pre-relativity notions which, although very useful, must first be discarded in order to appreciate the world according to GR. Specifically, we live in a world in which spacetime curvature is very small, and particles are for the most part moving quite slowly compared to the speed of light. Consequently, the mechanics of Galileo and Newton comes very naturally to us, even though it is only an approximation to the deeper story.

So we will set about learning the deeper story by gradually stripping away the layers of useful but misleading Newtonian intuition. The first step, which is the subject of this chapter, will be to explore special relativity (SR), the theory of spacetime in the absence of gravity (curvature). Hopefully this is mostly review, as it will proceed somewhat rapidly. The point will be both to recall what SR is all about, and to introduce tensors and related concepts that will be crucial later on, without the extra complications of curvature on top of everything else. Therefore, for this chapter we will always be working in flat spacetime, and furthermore we will only use inertial (Cartesian-like) coordinates. Needless to say it is possible to do SR in any coordinate system you like, but it turns out that introducing the necessary tools for doing so would take us halfway to curved spaces anyway, so we will put that off for a while.

1.2 ■ SPACE AND TIME, SEPARATELY AND TOGETHER

A purely cold-blooded approach to GR would reverse the order of Chapter 2 (Manifolds) and Chapter 1 (Special Relativity and Flat Spacetime). A *manifold* is the kind of mathematical structure used to describe spacetime, while *special relativity* is a model that invokes a particular kind of spacetime (one with no curvature, and hence no gravity). However, if you are reading this book you presumably have at least some familiarity with special relativity (SR), while you may not know anything about manifolds. So our first step will be to explore the relatively familiar territory of SR, taking advantage of this opportunity to introduce concepts and notation that will be crucial to later developments.

Special relativity is a theory of the structure of spacetime, the background on which particles and fields evolve. SR serves as a replacement for Newtonian mechanics, which also is a theory of the structure of spacetime. In either case, we can distinguish between this basic structure and the various dynamical laws govern-

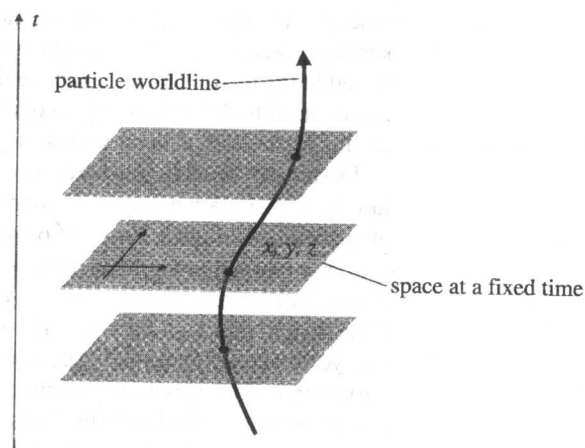


FIGURE 1.1 In Newtonian spacetime there is an absolute slicing into distinct copies of space at different moments in time. Particle worldlines are constrained to move forward in time, but can travel through space at any velocity; there is universal agreement on the question of whether two events at different points in space occur at the same moment of time.

ing specific systems: Newtonian gravity is an example of a dynamical system set within the context of Newtonian mechanics, while Maxwell's electromagnetism is a dynamical system operating within the context of special relativity.

Spacetime is a four-dimensional set, with elements labeled by three dimensions of space and one of time. (We'll do a more rigorous job with the definitions in the next chapter.) An individual point in spacetime is called an **event**. The path of a particle is a curve through spacetime, a parameterized one-dimensional set of events, called the **worldline**. Such a description applies equally to SR and Newtonian mechanics. In either case, it seems clear that "time" is treated somewhat differently than "space"; in particular, particles always travel forward in time, whereas they are free to move back and forth in space.

There is an important difference, however, between the set of allowed paths that particles can take in SR and those in Newton's theory. In Newtonian mechanics, there is a basic division of spacetime into well-defined slices of "all of space at a fixed moment in time." The notion of *simultaneity*, when two events occur at the same time, is unambiguously defined. Trajectories of particles will move ever forward in time, but are otherwise unconstrained; in particular, there is no limit on the relative velocity of two such particles.

In SR the situation is dramatically altered: in particular, *there is no well-defined notion of two separated events occurring "at the same time."* That is not to say that spacetime is completely structureless. Rather, at any event we can define a **light cone**, which is the locus of paths through spacetime that could conceivably be taken by light rays passing through this event. The absolute division, in Newtonian

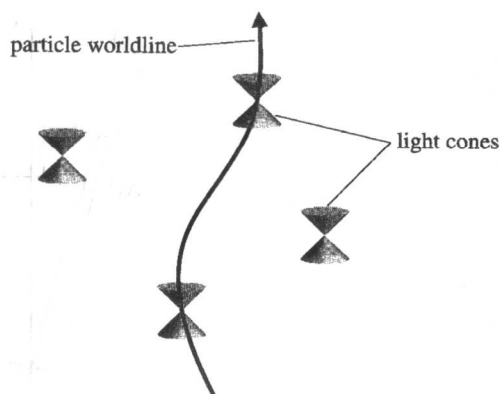


FIGURE 1.2 In special relativity there is no absolute notion of “all of space at one moment in time.” Instead, there is a rule that particles always travel at less than or equal to the speed of light. We can therefore define light cones at every event, which locally describe the set of allowed trajectories. For two events that are outside each others’ light cones, there is no universal notion of which event occurred earlier in time.

mechanics, of spacetime into unique slices of space parameterized by time, is replaced by a rule that says that physical particles cannot travel faster than light, and consequently move along paths that always remain inside these light cones.

The absence of a preferred time-slicing in SR is at the heart of why the notion of spacetime is more fundamental in this context than in Newtonian mechanics. Of course we can choose specific coordinate systems in spacetime, and once we do, it makes sense to speak of separated events occurring at the same value of the time coordinate in this particular system; but there will also be other possible coordinates, related to the first by “rotating” space and time into each other. This phenomenon is a natural generalization of rotations in Euclidean geometry, to which we now turn.

Consider a garden-variety two-dimensional plane. It is typically convenient to label the points on such a plane by introducing coordinates, for example by defining orthogonal x and y axes and projecting each point onto these axes in the usual way. However, it is clear that most of the interesting geometrical facts about the plane are independent of our choice of coordinates; there aren’t any preferred directions. As a simple example, we can consider the distance between two points, given by

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2. \quad (1.7)$$

In a different Cartesian coordinate system, defined by x' and y' axes that are rotated with respect to the originals, the formula for the distance is unaltered:

$$(\Delta s)^2 = (\Delta x')^2 + (\Delta y')^2. \quad (1.8)$$

We therefore say that the distance is invariant under such changes of coordinates.