中国科学院研究生教学丛书

现代概率论基础

FOUNDATIONS OF MODERN PROBABILITY

Olav Kallenberg

科学出版社 Springer

影印版

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内 容 简 介

本书属于中国科学院推荐的研究生原版教材之一，书中内容涉及十分广泛，从古典概率论原理、鞅、马尔可夫链到布朗运动、Lévy 过程、弱收敛、Itô 积分，再到现代概率论的发展如停时、半鞅、扩散过程等都做了详细介绍，堪称一部现代概率论的百科全书. 再配以由易到难的丰富的习题，非常适合作为硕士研究生和博士研究生概率论课程的教材。

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《中国科学院研究生教学丛书》序

在21世纪曙光初露,中国科技、教育面临重大改革和蓬勃发展之际,《中国科学院研究生教学丛书》——这套凝聚了中国科学院新老科学家、研究生导师们多年心血的研究生教材面世了。相信这套丛书的出版,会在一定程度上缓解研究生教材不足的困难,对提高研究生教育质量起着积极的推动作用。

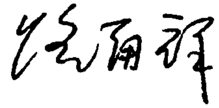
21世纪将是科学技术日新月异,迅猛发展的新世纪,科学技术将成为经济发展的最重要的资源和不竭的动力,成为经济和社会发展的首要推动力量。世界各国之间综合国力的竞争,实质上是科技实力的竞争。而一个国家科技实力的决定因素是它所拥有的科技人才的数量和质量。我国要想在21世纪顺利地实施"科教兴国"和"可持续发展"战略,实现邓小平同志规划的第三步战略目标——把我国建设成中等发达国家,关键在于培养造就一支数量宏大、素质优良、结构合理、有能力参与国际竞争与合作的科技大军。这是摆在我国高等教育面前的一项十分繁重而光荣的战略任务。

中国科学院作为我国自然科学与高新技术的综合研究与发展中心,在建院之初就明确了出成果出人才并举的办院宗旨,长期坚持走科研与教育相结合的道路,发挥了高级科技专家多、科研条件好、科研水平高的优势,结合科研工作,积极培养研究生;在出成果的同时,为国家培养了数以万计的研究生。当前,中国科学院正在按照江泽民同志关于中国科学院要努力建设好"三个基地"的指示,在建设具有国际先进水平的科学研究基地和促进高新技术产业发展基地的同时,加强研究生教育,努力建设好高级人

才培养基地，在肩负起发展我国科学技术及促进高新技术产业发展重任的同时，为国家源源不断地培养输送大批高级科技人才。

质量是研究生教育的生命，全面提高研究生培养质量是当前我国研究生教育的首要任务。研究生教材建设是提高研究生培养质量的一项重要的基础性工作。由于各种原因，目前我国研究生教材的建设滞后于研究生教育的发展。为了改变这种情况，中国科学院组织了一批在科学前沿工作，同时又具有相当教学经验的科学家撰写研究生教材，并以专项资金资助优秀的研究生教材的出版。希望通过数年努力，出版一套面向21世纪科技发展、体现中国科学院特色的高水平的研究生教学丛书。本丛书内容力求具有科学性、系统性和基础性，同时也兼顾前沿性，使阅读者不仅能获得相关学科的比较系统的科学基础知识，也能被引导进入当代科学研究的前沿。这套研究生教学丛书，不仅适合于在校研究生学习使用，也可以作为高校教师和专业研究人员工作和学习的参考书。

“桃李不言，下自成蹊。”我相信，通过中国科学院一批科学家的辛勤耕耘，《中国科学院研究生教学丛书》将成为我国研究生教育园地的一丛鲜花，也将似润物春雨，滋养莘莘学子的心田，把他们引向科学的殿堂，不仅为科学院，也为全国研究生教育的发展作出重要贡献。



Preface

Some thirty years ago it was still possible, as Loève so ably demonstrated, to write a single book in probability theory containing practically everything worth knowing in the subject. The subsequent development has been explosive, and today a corresponding comprehensive coverage would require a whole library. Researchers and graduate students alike seem compelled to a rather extreme degree of specialization. As a result, the subject is threatened by disintegration into dozens or hundreds of subfields.

At the same time the interaction between the areas is livelier than ever, and there is a steadily growing core of key results and techniques that every probabilist needs to know, if only to read the literature in his or her own field. Thus, it seems essential that we all have at least a general overview of the whole area, and we should do what we can to keep the subject together. The present volume is an earnest attempt in that direction.

My original aim was to write a book about "everything." Various space and time constraints forced me to accept more modest and realistic goals for the project. Thus, "foundations" had to be understood in the narrower sense of the early 1970s, and there was no room for some of the more recent developments. I especially regret the omission of topics such as large deviations, Gibbs and Palm measures, interacting particle systems, stochastic differential geometry, Malliavin calculus, SPDEs, measure-valued diffusions, and branching and superprocesses. Clearly plenty of fundamental and intriguing material remains for a possible second volume.

Even with my more limited, revised ambitions, I had to be extremely selective in the choice of material. More importantly, it was necessary to look for the most economical approach to every result I did decide to include. In the latter respect, I was surprised to see how much could actually be done to simplify and streamline proofs, often handed down through generations of textbook writers. My general preference has been for results conveying some new idea or relationship, whereas many propositions of a more technical nature have been omitted. In the same vein, I have avoided technical or computational proofs that give little insight into the proven results. This conforms with my conviction that the logical structure is what matters most in mathematics, even when applications is the ultimate goal.

Though the book is primarily intended as a general reference, it should also be useful for graduate and seminar courses on different levels, ranging from elementary to advanced. Thus, a first-year graduate course in measure-theoretic probability could be based on the first ten or so chapters, while the rest of the book will readily provide material for more advanced courses on various topics. Though the treatment is formally self-contained, as far as measure theory and probability are concerned, the text is intended for a rather sophisticated reader with at least some rudimentary knowledge of subjects like topology, functional analysis, and complex variables.

My exposition is based on experiences from the numerous graduate and seminar courses I have been privileged to teach in Sweden and in the United States, ever since I was a graduate student myself. Over the years I have developed a personal approach to almost every topic, and even experts might find something of interest. Thus, many proofs may be new, and every chapter contains results that are not available in the standard textbook literature. It is my sincere hope that the book will convey some of the excitement I still feel for the subject, which is without a doubt (even apart from its utter usefulness) one of the richest and most beautiful areas of modern mathematics.

Notes and Acknowledgments: My first thanks are due to my numerous Swedish teachers, and especially to Peter Jagers, whose 1971 seminar opened my eyes to modern probability. The idea of this book was raised a few years later when the analysts at Gothenburg asked me to give a short lecture course on "probability for mathematicians." Although I objected to the title, the lectures were promptly delivered, and I became convinced of the project's feasibility. For many years afterward I had a faithful and enthusiastic audience in numerous courses on stochastic calculus, SDEs, and Markov processes. I am grateful for that learning opportunity and for the feedback and encouragement I received from colleagues and graduate students.

Inevitably I have benefited immensely from the heritage of countless authors, many of whom are not even listed in the bibliography. I have further been fortunate to know many prominent probabilists of our time, who have often inspired me through their scholarship and personal example. Two people, Klaus Matthes and Gopi Kallianpur, stand out as particularly important influences in connection with my numerous visits to Berlin and Chapel Hill, respectively.

The great Kai Lai Chung, my mentor and friend from recent years, offered penetrating comments on all aspects of the work: linguistic, historical, and mathematical. My colleague Ming Liao, always a stimulating partner for discussions, was kind enough to check my material on potential theory. Early versions of the manuscript were tested on several groups of graduate students, and Kamesh Casukhela, Davorin Dujmovic, and Hussain Talibi in particular were helpful in spotting misprints. Ulrich Albrecht and Ed Slaminka offered generous help with software problems. I am further grateful to John Kimmel, Karina Mikhli, and the Springer production team for their patience with my last-minute revisions and their truly professional handling of the project.

My greatest thanks go to my family, who is my constant source of happiness and inspiration. Without their love, encouragement, and understanding, this work would not have been possible.

Olav Kallenberg
May 1997

Contents

Chapter 1

Elements of Measure Theory

σ-fields and monotone classes; measurable functions; measures and integration; monotone and dominated convergence; transformation of integrals; product measures and Fubini's theorem; L^p-spaces and projection; measure spaces and kernels

Modern probability theory is technically a branch of measure theory, and any systematic exposition of the subject must begin with some basic measure-theoretic facts. In this chapter we have collected some elementary ideas and results from measure theory that will be needed throughout this book. Though most of the quoted propositions may be found in any textbook in real analysis, our emphasis is often somewhat different and has been chosen to suit our special needs. Many readers may prefer to omit this chapter on their first encounter and return for reference when the need arises.

To fix our notation, we begin with some elementary notions from set theory. For subsets $A, A_k, B, \ldots$ of some abstract space Ω, recall the definitions of *union* $A \cup B$ or $\bigcup_k A_k$, *intersection* $A \cap B$ or $\bigcap_k A_k$, *complement* A^c, and *difference* $A \setminus B = A \cap B^c$. The latter is said to be *proper* if $A \supset B$. The *symmetric difference* of A and B is given by $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Among basic set relations, we note in particular the *distributive laws*

$$A \cap \bigcup\nolimits_k B_k = \bigcup\nolimits_k (A \cap B_k), \qquad A \cup \bigcap\nolimits_k B_k = \bigcap\nolimits_k (A \cup B_k),$$

and *de Morgan's laws*

$$\left\{\bigcup\nolimits_k A_k\right\}^c = \bigcap\nolimits_k A_k^c, \qquad \left\{\bigcap\nolimits_k A_k\right\}^c = \bigcup\nolimits_k A_k^c,$$

valid for arbitrary (not necessarily countable) unions and intersections. The latter formulas allow us to convert any relation involving unions (intersections) into the dual formula for intersections (unions).

A *σ-algebra* or *σ-field* in Ω is defined as a nonempty collection $\mathcal{A}$ of subsets of Ω such that $\mathcal{A}$ is closed under countable unions and intersections as well as under complementation. Thus, if $A, A_1, A_2, \ldots \in \mathcal{A}$, then also A^c, $\bigcup_k A_k$, and $\bigcap_k A_k$ lie in $\mathcal{A}$. In particular, the whole space Ω and the empty set $\emptyset$ belong to every σ-field. In any space Ω there is a smallest σ-field $\{\emptyset, \Omega\}$ and a largest one 2^Ω, the class of *all* subsets of Ω. Note that any σ-field $\mathcal{A}$ is closed under monotone limits. Thus, if $A_1, A_2, \ldots \in \mathcal{A}$ with $A_n \uparrow A$ or $A_n \downarrow A$, then also $A \in \mathcal{A}$. A *measurable space* is a pair $(\Omega, \mathcal{A})$, where Ω is a space and $\mathcal{A}$ is a σ-field in Ω.

For any class of σ-fields in Ω, the intersection (but usually not the union) is again a σ-field. If $\mathcal{C}$ is an arbitrary class of subsets of Ω, there is a smallest σ-field in Ω containing $\mathcal{C}$, denoted by $\sigma(\mathcal{C})$ and called the σ-field *generated* or *induced* by $\mathcal{C}$. Note that $\sigma(\mathcal{C})$ can be obtained as the intersection of all σ-fields in Ω that contain $\mathcal{C}$. A metric or topological space S will always be endowed with its *Borel σ-field* $\mathcal{B}(S)$ generated by the *topology* (class of open subsets) in S unless a σ-field is otherwise specified. The elements of $\mathcal{B}(S)$ are called *Borel sets*. In the case of the real line $\mathbb{R}$, we shall often write $\mathcal{B}$ instead of $\mathcal{B}(\mathbb{R})$.

More primitive classes than σ-fields often arise in applications. A class $\mathcal{C}$ of subsets of some space Ω is called a *π-system* if it is closed under finite intersections, so that $A, B \in \mathcal{C}$ implies $A \cap B \in \mathcal{C}$. Furthermore, a class $\mathcal{D}$ is a *λ-system* if it contains Ω and is closed under proper differences and increasing limits. Thus, we require that $\Omega \in \mathcal{D}$, that $A, B \in \mathcal{D}$ with $A \supset B$ implies $A \setminus B \in \mathcal{D}$, and that $A_1, A_2, \ldots \in \mathcal{D}$ with $A_n \uparrow A$ implies $A \in \mathcal{D}$.

The following *monotone class theorem* is often useful to extend an established property or relation from a class $\mathcal{C}$ to the generated σ-field $\sigma(\mathcal{C})$. An application of this result is referred to as a *monotone class argument.*

Theorem 1.1 *(monotone class theorem, Sierpiński) Let $\mathcal{C}$ be a π-system and $\mathcal{D}$ a λ-system in some space Ω such that $\mathcal{C} \subset \mathcal{D}$. Then $\sigma(\mathcal{C}) \subset \mathcal{D}$.*

Proof: We may clearly assume that $\mathcal{D} = \lambda(\mathcal{C})$, the smallest λ-system containing $\mathcal{C}$. It suffices to show that $\mathcal{D}$ is a π-system, since it is then a σ-field containing $\mathcal{C}$ and therefore must contain the smallest σ-field $\sigma(\mathcal{C})$ with this property. Thus, we need to show that $A \cap B \in \mathcal{D}$ whenever $A, B \in \mathcal{D}$.

The relation $A \cap B \in \mathcal{D}$ is certainly true when $A, B \in \mathcal{C}$, since $\mathcal{C}$ is a π-system contained in $\mathcal{D}$. The result may now be extended in two steps. First we fix an arbitrary set $B \in \mathcal{C}$ and define $\mathcal{A}_B = \{A \subset \Omega;\ A \cap B \in \mathcal{D}\}$. Then $\mathcal{A}_B$ is a λ-system containing $\mathcal{C}$, and so it contains the smallest λ-system $\mathcal{D}$ with this property. This shows that $A \cap B \in \mathcal{D}$ for any $A \in \mathcal{D}$ and $B \in \mathcal{C}$. Next fix an arbitrary set $A \in \mathcal{D}$, and define $\mathcal{B}_A = \{B \subset \Omega;\ A \cap B \in \mathcal{D}\}$. As before, we note that even $\mathcal{B}_A$ contains $\mathcal{D}$, which yields the desired property. □

For any family of spaces Ω_t, $t \in T$, we define the *Cartesian product* $\mathsf{X}_{t\in T}\Omega_t$ as the class of all collections $(\omega_t;\ t \in T)$, where $\omega_t \in \Omega_t$ for all t. When $T = \{1, \ldots, n\}$ or $T = \mathbb{N} = \{1, 2, \ldots\}$, we shall often write the product space as $\Omega_1 \times \cdots \times \Omega_n$ or $\Omega_1 \times \Omega_2 \times \cdots$, respectively, and if $\Omega_t = \Omega$ for all t, we shall use the notation Ω^T, Ω^n, or Ω^∞. In case of topological spaces Ω_t, we endow $\mathsf{X}_t\Omega_t$ with the product topology unless a topology is otherwise specified.

Now assume that each space Ω_t is equipped with a σ-field $\mathcal{A}_t$. In $\mathsf{X}_t\Omega_t$ we may then introduce the *product σ-field* $\bigotimes_t \mathcal{A}_t$, generated by all one-dimensional *cylinder sets* $A_t \times \mathsf{X}_{s\neq t}\Omega_s$, where $t \in T$ and $A_t \in \mathcal{A}_t$. (Note the analogy with the definition of product topologies.) As before, we shall write $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$, $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots$, $\mathcal{A}^T$, $\mathcal{A}^n$, or $\mathcal{A}^\infty$ in the appropriate special cases.

Lemma 1.2 *(product and Borel σ-fields) Let $S_1, S_2, \ldots$ be separable metric spaces. Then*

$$\mathcal{B}(S_1 \times S_2 \times \cdots) = \mathcal{B}(S_1) \otimes \mathcal{B}(S_2) \otimes \cdots .$$

Thus, for countable products of separable metric spaces, the product and Borel σ-fields agree. In particular, $\mathcal{B}(\mathbb{R}^d) = (\mathcal{B}(\mathbb{R}))^d = \mathcal{B}^d$, the σ-field generated by all rectangular boxes $I_1 \times \cdots \times I_d$, where $I_1, \ldots, I_d$ are arbitrary real intervals.

Proof: The assertion may be written as $\sigma(\mathcal{C}_1) = \sigma(\mathcal{C}_2)$, and it suffices to show that $\mathcal{C}_1 \subset \sigma(\mathcal{C}_2)$ and $\mathcal{C}_2 \subset \sigma(\mathcal{C}_1)$. For $\mathcal{C}_2$ we may choose the class of all cylinder sets $G_k \times \mathsf{X}_{n \neq k} S_n$ with $k \in \mathbb{N}$ and G_k open in S_k. Those sets generate the product topology in $S = \mathsf{X}_n S_n$, and so they belong to $\mathcal{B}(S)$.

Conversely, we note that $S = \mathsf{X}_n S_n$ is again separable. Thus, for any topological base $\mathcal{C}$ in S, the open subsets of S are countable unions of sets in $\mathcal{C}$. In particular, we may choose $\mathcal{C}$ to consist of all finite intersections of cylinder sets $G_k \times \mathsf{X}_{n \neq k} S_n$ as above. It remains to note that the latter sets lie in $\bigotimes_n \mathcal{B}(S_n)$. □

Every point mapping f between two spaces S and T induces a set mapping f^{-1} in the opposite direction, that is, from 2^T to 2^S, given by

$$f^{-1}B = \{s \in S;\ f(s) \in B\}, \quad B \subset T.$$

Note that f^{-1} preserves the basic set operations in the sense that for any subsets B and B_k of T,

$$f^{-1}B^c = (f^{-1}B)^c, \quad f^{-1}\bigcup_k B_k = \bigcup_k f^{-1}B_k, \quad f^{-1}\bigcap_k B_k = \bigcap_k f^{-1}B_k. \tag{1}$$

The next result shows that f^{-1} also preserves σ-fields, in both directions. For convenience we write

$$f^{-1}\mathcal{C} = \{f^{-1}B;\ B \in \mathcal{C}\}, \quad \mathcal{C} \subset 2^T.$$

Lemma 1.3 *(induced σ-fields) Let f be a mapping between two measurable spaces $(S, \mathcal{S})$ and $(T, \mathcal{T})$. Then $f^{-1}\mathcal{T}$ is a σ-field in S, whereas $\{B \subset T;\ f^{-1}B \in \mathcal{S}\}$ is a σ-field in T.*

Proof: Use (1). □

Given two measurable spaces $(S, \mathcal{S})$ and $(T, \mathcal{T})$, a mapping $f : S \to T$ is said to be $\mathcal{S}/\mathcal{T}$*-measurable* or simply *measurable* if $f^{-1}\mathcal{T} \subset \mathcal{S}$, that is, if $f^{-1}B \in \mathcal{S}$ for every $B \in \mathcal{T}$. (Note the analogy with the definition of continuity in terms of topologies on S and T.) By the next result, it is enough to verify the defining condition for a generating subclass.