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Stability of Numerical Methods for Delay Differential Equations

(延时微分方程数值方法的稳定性)

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Preface

Delay differential equations (DDEs) are also known as functional differential equations or differential-difference equations. They play an important role in the research of various applied sciences, such as control theory, population dynamics, electrical networks, environment science, biology, bioecology, and life science. Delay differential equations are often used as their mathematical models. As we know, the subject of ordinary differential equations is as old as the subject of calculus. Although the subject of delay differential equations is relatively young, its theoretical study has been matured today. However, the field of the numerical methods for delay differential equations is still new. It was not systematically studied until 1975. The reason may be that in the early years researchers considered the numerical treatment of delay differential equations similar to that of ordinary differential equations and unnecessary to pay special attention to delay differential equations. Contrary to the common belief that time, the numerical treatment of delay differential equations is much more complicated than that of ordinary differential equations. The reason is that in those equations, the instantaneous derivative of the unknown function depends not only on the time and the value of the unknown function at the time, but also the values of the unknown function (even its derivative) prior to the time. This leads to the complication of the difference equations after discretization. Consequently, solving such difference equations is very difficult. For example, even a direct discretization of a simple pantograph equation results in a variable order and variable coefficient linear difference equation.

Internationally, before 1975, there were few publications on the study of delay differential equations, for example, Zverkina in 1960, Miranker in 1962, Snow in 1965, Feldstein in 1969, and Javernini in 1971. In 1975, Barwell [2, 3] proposed the concepts of P-stability and GP-stability, which are similar to the A-stability in the numerical ordinary differential equations. In 1984, Bellen studied the one-step collocation methods for delay differential equations. In the same year, Jackiewicz [25] discussed the asymptotic stability of the θ-methods. For the Runge-Kutta methods, in 1986, Zennaro [61] studied the P-stability properties. In 1988, Bellen, Jackiewicz, and Zennaro [4] discussed the asymptotic stability of the implicit Runge-Kutta methods for neutral delay differential equations. In 1985, Watanabe and Roth [58] were the first to study the P-stability and GP-stability of the linear multistep methods. At Leiden University in Holland, In't Hout, M.Z. Liu, and Spijker in 1990 [45], 1991 [24], 1992 [21, 20], and 1993 [44] made outstanding contributions to the study of the P-stability and GP-stability. Torelli in 1989 [55] and 1991 [56] studied nonlinear delay differential equations and proposed the concepts of GPN-stability and GRN-stability. These concepts correspond to the AN-stability and BN-stability in ordinary differential equations. For neutral equations, in 1994, Tian et al first proposed the concept of NGP-stability [41]. In 1997, Kuang [35] gave another important concept of PL-stability corresponding to the L-stability in ordinary differential equations. For the generalized neutral equations, Cong [14, 13, 11] gave the concept of the NGP_G-stability and presented sufficient and necessary conditions for the NGP_G-stability of the implicit Runge-Kutta methods and linear multistep methods. Other authors of the study of numerical stability of delay differential equations and neutral delay differential equations include Koto [27, 28, 29, 30, 31], Qiu and Mitsui [49, 50]. In recent years, the study of pantograph equations has been a hot topic. Various concepts of stability have been proposed. The reader can find more terms of stability in the index.

In addition to an introduction to the numerical methods for ordinary differential equations, this book emphasizes the study of the stability of the difference equations obtained by discretizing various kinds of delay differential equations. The reason for this emphasis is that for a numerical method the only issue other than convergence (including accuracy) is its stability. Unstable methods or methods with very small stability regions require extreme caution, because rounding errors may overwhelm numerical solutions, even halt computer programs. Whereas methods with good stability maintain the asymptotic or instantaneous behaviour of theoretical (analytic) solutions. This is a desirable property for engineers. Because of numerous kinds of delay differential equations, there are a number of concepts of stability. The purpose of this book is to introduce the basic concepts and theory of the stability of the numerical methods for solving delay differential equations and basic techniques for proving stability of numerical methods. So the reader can apply the techniques elsewhere.

The first three chapters of this book review the basic methods for solving the initial value problems of ordinary differential equations and the analysis of stability. Chapter 4 introduces the numerical methods for the linear test equations, discusses the P-stability and GP-stability. Chapter 5 generalizes the results in Chapter 4 to systems of delay differential equations. Chapter 6 discusses the GPN-stability and GRN-stability of the Runge-Kutta methods for the nonautonomous linear and nonlinear differential equations respectively. Chapter 7 studies the *NGP*-stability of neutral delay differential equations. Chapter 8 discusses the reducible quadratures for solving the second class Volterra delay integral equations and their numerical stability. Chapter 9 studies the numerical methods for the pantograph delay differential equations and their stability analysis.

Starting Chapter 4, at the end of each chapter, we give a big picture outlining the content of the chapter. This book is based on the references listed in Bibliography. The supplementary references closely related to this book are listed in Suggestion for Further Reading.

This book is for graduate students in mathematics, physics, and engineering. It can also be used as a reference book for researchers or teachers in related areas. The prerequisites for this book are calculus, complex analysis, matrix theory, and basic knowledge of ordinary and delay differential equations.

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Authors
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Contents

Chapter 1

Linear Multistep Methods

In this chapter, we study the fundamental concepts and theory of linear multistep methods for solving ordinary differential equations (ODEs). First, we motivate the study of the stability of the numerical methods for delay differential equations (DDE) in Section 1.1. Then, in Section 1.2, we present the concepts of convergence, consistency, and stability and their relations. In Section 1.3, we answer the question: What is the maximal order a zero-stable linear multistep method can attain? Finally, Section 1.4 is devoted to the topic of the A-stability.

1.1 Introduction

In an ordinary differential equation (ODE)

$$y'(t) = f(t, y(t))$$

the derivative $y'(t)$ is a function of time t and $y(t)$. In many applications, the derivative, the rate of change, at time t depends not only on t and $y(t)$ but also the solution function evaluated at an earlier time $y(t-\tau)$, where $\tau > 0$. Thus a delay differential equation (DDE) has the form:

$$y'(t) = f(t, y(t), y(t-\tau)).$$

In early years, it was believed that solving DDEs, or functional differential equations in general, was not different from solving ODEs and it was unnecessary to pay special attention to the study of DDEs. On the contrary, using the linear multistep methods or the Runge-Kutta methods, the stability analysis of numerical methods for DDEs is much more complicated than that for ODEs. In the following, using a general linear multistep method, we illustrate the difference between DDE and ODE. Consider the linear k-step method:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \qquad \alpha_k \neq 0 \quad \text{and} \quad \alpha_0^2 + \beta_0^2 \neq 0 \tag{1.1}$$

for solving the initial value problem of the single first-order ODE

$$\begin{cases} y'(t) = f(t, y(t)), & t \geqslant t_0, \\ y(t_0) = \eta, \end{cases} \tag{1.2}$$

where f is a complex-valued function and sufficiently differentiable guaranteeing the existence of a unique solution of the initial value problem (1.2). To study the stability of the numerical method, applying the method (1.1) to the test problem

$$\begin{cases} y'(t) = \lambda y(t), & \mathrm{Re}(\lambda) < 0, \\ y(t_0) = \eta, \end{cases}$$

we obtain the recurrence relation

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\lambda \sum_{j=0}^{k} \beta_j y_{n+j}. \tag{1.3}$$

The above equation is also called the linear difference equation for $\{y_n\}$. From the theory of linear difference equations, the solution $\{y_n\}$ of the difference equation (1.3) satisfies

$$\lim_{n\to\infty} y_n = 0$$

for any initial values $y_0, y_1, \cdots, y_{k-1}$ if and only if the characteristic polynomial

$$p(z) = \rho(z) - \bar{h}\sigma(z), \qquad \bar{h} = \lambda h, \quad h > 0, \tag{1.4}$$

is a Schur polynomial, where $\rho(z) = \sum_{j=0}^{k} \alpha_j z^j$ and $\sigma(z) = \sum_{j=0}^{k} \beta_j z^j$. In other words, all the roots of the characteristic polynomial (1.4) lie inside the unit circle. To determine whether $p(z)$ is a Schur polynomial, since the degree of $p(z)$ is a constant, we repeatedly reduce the degree of $p(z)$ by using the Schur criterion and finally determine whether a polynomial of low degree is a Schur polynomial. Specifically, let

$$\begin{aligned} p(z) &= c_k z^k + c_{k-1} z^{k-1} + \cdots + c_1 z + c_0, \\ \bar{p}(z) &= c_0^* z^k + c_1^* z^{k-1} + \cdots + c_{k-1}^* z + c_k^*, \end{aligned}$$

where c_i^* are the complex conjugate of c_i $(i = 0, 1, \cdots, k)$. The Schur criterion says that $p(z)$ is a Schur polynomial if and only if

$$p_1(z) = \frac{1}{z}[\bar{p}(0)p(z) - p(0)\bar{p}(z)]$$

is a Schur polynomial and $|\bar{p}(0)| > |p(0)|$. Obviously, $p_1(z)$ is of degree $k-1$. Repeating the above procedure, we only determine whether a polynomial of low degree is a Schur polynomial.

Now, to show the difference between ODE and DDE, we apply the linear multistep method (1.1) to the following simple model DDE:

$$\begin{cases} y'(t) = ay(t) + by(t-\tau), & t \geqslant 0, \\ y(t) = \phi(t), & t \leqslant 0, \end{cases}$$

where $\tau > 0$ and $|b| < -\mathrm{Re}(a)$. Let $h = \tau/m$, where $m \geqslant 1$ is an integer, be the step size in the method (1.1), then we have the difference equation

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = ha \sum_{j=0}^{k} \beta_j y_{n+j} + hb \sum_{j=0}^{k} \beta_j y_{n+j-m}.$$

Its characteristic polynomial is (see also Section 4.3)

$$p_m(z) = Q(z)z^m + p(z),$$

where

$$\begin{aligned} Q(z) &= \rho(z) - \hat{a}\,\sigma(z), \\ p(z) &= \sigma(z)\hat{b}, \\ \hat{a} &= ha, \\ \hat{b} &= hb. \end{aligned}$$

Because $m \geqslant 1$ is an arbitrary positive integer, the Schur criterion cannot be used to determine whether $p_m(z)$ is a Schur polynomial. We must find other ways. This clearly shows that the problems arisen from DDEs are more complicated than those from ODEs.

1.2 Consistency, Convergence and Stability

Before studying the stability of a linear multistep method, we introduce the definitions of convergence and zero-stability, related concepts, and their relations.

Definition 1.2.1 (convergence) *Suppose that the solution of the initial value problem (1.2) uniquely exists and the solution sequence obtained by applying the linear multistep method (1.1) to the problem (1.2) is $\{y_n\}$, where the initial values $y_0, y_1, \cdots, y_{k-1}$ satisfy*

$$\lim_{h\to 0} y_i = \lim_{h\to 0} \eta_i(h) = \eta, \quad i = 0, 1, \cdots, k-1,$$

then the linear multistep method (1.1) is said to be convergent *if*

$$\lim_{h\to 0} y_n = y(t_n), \quad nh = t_n - t_0$$

holds for any $t_n \in [t_0, b]$.

Example Apply the Euler's method

$$y_{n+1} = y_n + hf_n, \quad f_n = f(t_n, y_n)$$

to the test equation

$$\begin{cases} y'(t) = \lambda\, y(t), \\ y(0) = 1. \end{cases}$$

Let $t = nh$ be fixed, then

$$\begin{aligned} y_n &= y_{n-1} + hf_{n-1} \\ &= (1 + \lambda h) y_{n-1} \\ &= \left(1 + \frac{\lambda t}{n}\right)^n y_0 \\ &= \left(1 + \frac{\lambda t}{n}\right)^n \eta_0(h) \to e^{\lambda t}, \quad \text{as } h \to 0, \end{aligned}$$

which shows that the Euler's method is convergent.

Now we turn to the concept of the order, which indicates the accuracy, of a numerical method.

Suppose that $y(t) \in C^1[t_0, b]$. Define the difference operator

$$\mathcal{L}[y(t); h] \equiv \sum_{j=0}^{k} [\alpha_j y(t + jh) - h\beta_j y'(t + jh)] \tag{1.5}$$

associated with the linear multistep method (1.1). Using the Taylor expansions of $y(t + jh)$ and $y'(t + jh)$ about t

$$\begin{aligned} y(t + jh) &= y(t) + jhy'(t) + \frac{(jh)^2}{2!} y''(t) + \cdots \\ y'(t + jh) &= y'(t) + jhy''(t) + \frac{(jh)^2}{2!} y^{(3)}(t) + \cdots \end{aligned}$$

and collecting the terms of the same power of h, from (1.5), we get

$$\mathcal{L}[y(t); h] = c_0 y(t) + c_1 h y'(t) + \cdots + c_q h^q y^{(q)}(t) + \cdots, \tag{1.6}$$

where

$$\begin{aligned} c_0 &= \alpha_0 + \alpha_1 + \cdots + \alpha_k, \\ c_1 &= \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k - (\beta_0 + \beta_1 + \cdots + \beta_k), \\ &\vdots \\ c_q &= \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + \cdots + k^q\alpha_k) - \frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2 + \cdots + k^{q-1}\beta_k). \end{aligned} \tag{1.7}$$

Definition 1.2.2 (order) *The difference operator $\mathcal{L}$ in (1.5) or the associated linear multistep method $\langle \rho, \sigma \rangle$ is said to be of* order p, *if $c_0 = c_1 = \cdots = c_p = 0$ and $c_{p+1} \neq 0$ in (1.6). We call $c_{p+1}/\sigma(1)$ the* error constant *of the method (1.1).*

Definition 1.2.3 (consistency) *If the linear multistep method (1.1) is of order $p \geqslant 1$, then it is said to be* consistent.

Obviously, from Definition 1.2.2 and (1.7), the method (1.1) is consistent if and only if

$$c_0 = c_1 = 0$$

or

$$\rho(1) = 0 \quad \text{and} \quad \rho'(1) = \sigma(1).$$

Associated with the difference operator $\mathcal{L}$, we define truncation errors.

Definition 1.2.4 (local truncation error) *The local truncation error of the linear k-step method (1.1) is defined by*

$$T_{n+k} \equiv \mathcal{L}[y(t_n); h],$$

where $\mathcal{L}$ is given by (1.5) and $y(t)$ is the theoretical solution of the initial value problem (1.2).

The truncation error T_{n+k} is local in the following sense. Under the localization assumption

$$y_{n+j} = y(t_{n+j}), \quad j = 0, 1, \cdots, k-1,$$

for the linear multistep method (1.1), that is, there are no previous truncation errors, the application of (1.1) to (1.2) yields y_{n+k} and shows that the local truncation error T_{n+k} is proportional to $y(t_{n+k}) - y_{n+k}$. Without the localization assumption, $y(t_{n+k}) - y_{n+k} \equiv e_{n+k}$ is called the *global truncation error*.

Now we give the concept of zero-stability, recalling that $\rho(z)$ and $\sigma(z)$ denote respectively the first and second characteristic polynomials.

Definition 1.2.5 (zero-stability) *Let ξ_j, $j = 1, \cdots, k$, be the roots of the first characteristic polynomial $\rho(\xi)$ of the linear k-step method (1.1). If $|\xi_j| \leqslant 1$ for $j = 1, \cdots, k$ and for any j_0 such that $|\xi_{j_0}| = 1$, ξ_{j_0} must be a single root of $\rho(\xi)$, then the method (1.1) is said to be zero-stable (or Dahlquist stable or that the method (1.1) satisfies the* root condition*).*

Now that we have introduced concepts, in the rest of this section we present the relations between convergence, consistency, and zero-stability.

From the Definition 1.2.1 of convergence, for a method to be useful, it should be convergent. The following theorem, Dahlquist's fundamental theorem states that a convergent linear multistep method is zero-stable.

Theorem 1.2.1 *If the linear multistep method (1.1) is convergent, then it is zero-stable.*

Proof. Consider the initial value problem

$$y'(t) = 0, \quad y(0) = 0.$$

Its theoretical solution $y(t) \equiv 0$. Applying (1.1) to the above problem, we have the difference equation

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = 0,$$

whose characteristic polynomial is

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j.$$

We first consider the case when all the roots $\xi_1, \xi_2, \cdots, \xi_k$ of $\rho(\xi)$ are distinct. Then the general solution y_n of the difference equation satisfying the requirement that the starting values $y_m \to y(0)$ as $h \to 0$, $m = 0, 1, \cdots, k-1$, has the form

$$y_n = h(d_1 \xi_1^n + d_2 \xi_2^n + \cdots + d_k \xi_k^n),$$

where d_i $(i = 1, 2, \cdots, k)$ are arbitrary constants. Since d_i are arbitrary, $y_n \to 0$ $(n \to \infty)$ if and only if

$$\lim_{\substack{h \to 0i \\ nh=t}} h\xi_i^n = 0,$$

for each i, that is

$$\lim_{\substack{h\to 0\\ nh=t}} t\frac{\xi_i^n}{n} = 0 \quad \text{if and only if} \quad |\xi_i| \leqslant 1.$$

We then consider the case when ξ_i is a multiple root of $\rho(\xi)$ with multiplicity $q > 1$. Then the contribution of ξ_i to the solution y_n is of the form

$$h[d_{i,1} + d_{i,2}n + d_{i,3}n(n-1) + \cdots + d_{i,q}n(n-1)\cdots(n-q+2)]\xi_i^n.$$

Thus the general solution has a term $d_i hn^{q-1}\xi_i^n$ and for $q > 1$,

$$\lim_{\substack{h\to 0\\ nh=t}} hn^{q-1}\xi_i^n = \lim_{\substack{h\to 0\\ nh=t}} tn^{q-2}\xi_i^n = 0 \quad \text{if and only if} \quad |\xi_i| < 1.$$

This completes the proof. □

The following theorem states another condition for a convergent linear multistep method.

Theorem 1.2.2 *A convergent linear multistep method (1.1) is necessarily consistent.*

Proof. We first prove that $c_0 = 0$. Consider the initial value problem

$$\begin{cases} y'(t) = 0, \\ y(0) = 1, \end{cases}$$

whose theoretical solution $y(t) = 1$. Applying (1.1) to the above problem, we get

$$\alpha_0 y_n + \alpha_1 y_{n+1} + \cdots + \alpha_k y_{n+k} = 0. \tag{1.8}$$

Assume that $y_m = 1$ for $m = 0, 1, \cdots, k-1$. Since the linear multistep method is convergent,

$$\lim_{\substack{h\to 0\\ nh=t}} y_n = 1.$$

Letting $n \to \infty$ in (1.8), we have

$$\alpha_0 + \alpha_1 + \cdots + \alpha_k = 0,$$

that is $c_0 = 0$.

Next, to prove $c_1 = 0$, we consider the initial value problem

$$\begin{cases} y'(t) = 1, \\ y(0) = 0, \end{cases}$$

whose theoretical solution $y(t) = t$. Applying (1.1) to this problem, we get

$$\alpha_0 y_n + \alpha_1 y_{n+1} + \cdots + \alpha_k y_{n+k} = h(\beta_0 + \beta_1 + \cdots + \beta_k). \tag{1.9}$$

It can be verified that $y_n = nh\kappa$, where

$$\kappa = (\beta_0 + \beta_1 + \cdots + \beta_k)/(k\alpha_k + \cdots + \alpha_1),$$

is a solution sequence of (1.9). It follows from the convergence of the method (1.1) that

$$\lim_{\substack{h \to 0 \\ nh=t}} y_n = t,$$

implying $t\kappa = t$. Thus $\kappa = 1$, that is $\rho'(1) = \sigma(1)$ or $c_1 = 0$. □

Before presenting relations between consistency and zero-stability, we introduce the concept of stability related to zero-stability.

Definition 1.2.6 (stability) *Let $\{y_n\}$ and $\{z_n\}$ be two solution sequences of (1.1). If for any $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\max_{t_0 \leqslant t_n \leqslant b} |z_n - y_n| \leqslant \varepsilon$$

when

$$\max_{0 \leqslant j \leqslant k-1} |z_j - y_j| \leqslant \delta, \quad 0 < h \leqslant h_0,$$

then we say that the method (1.1) is stable.

The following theorem establishes that stable and zero-stable are equivalent under the condition of consistency.

Theorem 1.2.3 *If the linear multistep method (1.1) is consistent, then it is stable if and only if it is zero-stable.*

Proof. To prove this theorem, we consider the initial value problem

$$\begin{cases} y'(t) = 0, \\ y(0) = 0, \end{cases}$$

whose theoretical solution is $y(t) = 0$. Applying (1.1) to the above problem with the starting values $y_0 = y_1 = \cdots = y_{k-1} = 0$, we get

$$y_n = 0, \quad \forall n \in N.$$

For the starting values $z_0 = \varepsilon,\ z_1 = \varepsilon\,\xi_j,\ \cdots,\ z_{k-1} = \varepsilon\,\xi_j^{k-1}$, we have the solution

$$z_n = \varepsilon \xi_j^n, \quad \forall n \in N.$$

Let ξ_j, $|\xi_j| > 1$, be a root of $\rho(\xi)$, then

$$\max_{0 \leqslant t_n \leqslant b} |y_n - z_n| \leqslant \varepsilon\, |\xi_j|^{N(h)},$$

where $N(h)$ is the largest positive integer n such that $nh \leqslant b$. Obviously, $N(h) \to \infty$ when $h \to 0$. It then follows that

$$\max_{0 \leqslant t_n \leqslant b} |y_n - z_n| \to \infty \quad \text{as} \quad h \to 0,$$

which shows that the method (1.1) is not stable. For a multiple root ξ_j, $|\xi_j| = 1$, the proof is similar.

Next, suppose that (1.1) is zero-stable, we prove that the method is stable. For simplicity, we prove this for the equation $y' = \lambda y$. We also assume that the roots of $\rho(\xi)$ are distinct. Applying (1.1) to the equation $y' = \lambda y$, we get

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\lambda \sum_{j=0}^{k} \beta_j y_{n+j}. \tag{1.10}$$

Let $e_n = y_n - z_n$ and $|e_n| \leqslant \varepsilon$, $n = 0, 1, \cdots, k-1$. From (1.10), we have

$$\sum_{j=0}^{k} (\alpha_j - h\lambda\beta_j) e_{n+j} = 0. \tag{1.11}$$

For sufficiently small $h > 0$, the roots $\xi_1(\bar{h}), \xi_2(\bar{h}), \cdots, \xi_k(\bar{h})$, where $\bar{h} = h\lambda$, are distinct and $\xi_i = \xi_i(0)$ $(i = 1, 2, \cdots, k)$. Thus the general solution of (1.11) can be expressed as

$$e_n = \sum_{j=0}^{k} r_j [\xi_j(\bar{h})]^n, \quad n \geqslant 0. \tag{1.12}$$

Let $n = 0, 1, \cdots, k-1$, then we have the system

$$\begin{aligned} e_0 &= r_1 + r_2 + \cdots + r_k, \\ e_1 &= r_1 \xi_1(\bar{h}) + r_2 \xi_2(\bar{h}) + \cdots + r_k \xi_k(\bar{h}), \\ &\ \vdots \\ e_{k-1} &= r_1 [\xi_1(\bar{h})]^{k-1} + r_2 [\xi_2(\bar{h})]^{k-1} + \cdots + r_k [\xi_k(\bar{h})]^{k-1}. \end{aligned}$$