GLOBAL ANALYSIS
Pure and Applied

Manifolds, Tensor Analysis, and Applications

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Foreword

AIM AND SCOPE OF THE SERIES

What Is Global Analysis?

From ancient times till Newton, mathematics meant geometry and algebra. Then analysis (now called classical) was born, along with the foundations of physics, engineering, and modern science. Among the outstanding events of modern mathematics are the syntheses of these fields along common frontiers. The synthesis of classical analysis and geometry is now called *global analysis*.

The History of Global Analysis and Its Applications

Important pioneers in the synthesis of global analysis were Henri Poincaré (1880s), George Birkhoff (1920s), Marston Morse (1930s), and Hassler Whitney (1940s). The technical tools of differential topology (1950s) made the final synthesis possible (1960s). Through the efforts of Solomon Lefshetz (1950s), the work of the Russian school (Liapounov, Andronov, Pontriagin) on dynamics became widely known in the west and included in this synthesis. A veritable explosion of new results and applications followed in the 1970s.

From the earliest work of Poincaré and Liapounov onward, the applications of geometry and analysis to astronomy, physics, and engineering provided the explicit motivation for much of this work. The current form of the theory reflects this pervasive influence in its direct applicability to these fields. It has already created new and powerful methods of applied mathematics, which complement existing tools such as perturbation methods, asymptotics, and numerical techniques.

Far from being the exclusive preserve of pure mathematicians, global analysis has its roots in physical problems and can be redirected to these problems once again, often with startling results.

Target of the Series: The Accessibility of Global Analysis

There is a great contrast between the potential importance of global analysis and the great difficulty of learning about it. A growing number of scientists of all disciplines have discovered that the techniques of global analysis have important applications in their own fields; they are looking seriously for keys to these techniques. This series will attempt to provide the keys.

Needed are books that introduce the basic concepts and their applications, texts that develop the prerequisites for more serious study accessibly and compactly, and advanced monographs which make the research frontier available to a wide audience of scientists and engineers who have acquired these prerequisites. To these ends, this series will deal with such subjects as

Theory
- Linear algebra and representation theory
- Calculus on manifolds and bundles
- Differential geometry and Lie theory
- Manifolds of mappings and sections
- Transversal approximations
- Calculus of variations in the large
- Dynamical systems theory and nonlinear oscillations
- Nonlinear actions of Lie groups

Applications
- Classical mechanics and field theory
- Geometric quantization
- Hydrodynamics
- Elastomechanics
- Econometrics
- Social theory
- Morphogenesis
- Network theory

and other topics of pure and applied global analysis.

AUDIENCES

Sub-Series A. The advanced texts will provide reports on theory or applications from the research frontier in expository style for specialists, or for nonspecialists who have the prerequisite mathematical background. For example, graduate students of science or engineering as well as mathematics will find them manageable.

Sub-Series B. The basic texts will provide a complete curriculum of essential prerequisites, starting with advanced linear algebra and calculus, for the advanced texts of Sub-Series A. These texts will be suitable for advanced undergraduate courses in pure and applied mathematics, or as reference works for research in engineering, the sciences, or mathematics.

UNIQUE FEATURES

Through the basic texts (Sub-Series B) covering all the prerequisites in a uniform style and the advanced texts (Sub-Series A) building on this foundation, it will be possible for readers to study the detailed applications of global analysis to their own fields (as they appear in the series), to form independent evaluations of the new methods, and to master the techniques for their own use if it is justified.

Sub-Series B starts from the post-calculus level, in textbook format with worked examples, exercises and adequate illustrations. The series will give a complete library of prerequisites, together with new contributions to global analysis and some outstanding examples of its applications, illustrating the new methods in applied mathematics. All the texts will be in English and conform as far as possible to a common notational scheme.

RALPH ABRAHAM
PHILIP J. HOLMES
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PREFACE

The purpose of this book is to provide core background material in global analysis for mathematicians sensitive to applications and to physicists, engineers, and mathematical biologists. The main goal is to provide a working knowledge of manifolds, dynamical systems, tensors, and differential forms. Some applications to Hamiltonian mechanics, fluid mechanics, electromagnetism, and control theory are given in Chapter 8, using both invariant and index notation. Detailed treatments of these and additional applications are planned for other volumes in the series. The book does not deal with Riemannian geometry in detail or with Lie groups or Morse theory. These too are planned for a subsequent volume.

Throughout the text special or supplementary topics occur in boxes. This device enables the reader to skip various topics without disturbing the main flow of the text. Additional background material in the appendices is given for completeness, to minimize the necessity of consulting too many outside references.

We treat finite and infinite-dimensional manifolds simultaneously. This is partly for efficiency of exposition. Without advanced applications, using, say, manifolds of mappings, the study of infinite-dimensional manifolds is hard to motivate, except for its intrinsic interest. Chapter 8 gives a hint of these applications. In fact, some readers may wish to skip the infinite-dimensional case altogether. To aid in this we have separated into boxes many of the technical points peculiar to the infinite-dimensional case.

Our own research interests lean heavily toward physical applications, and the choice of topics is partly molded by what has been and remains useful for this kind of research. Some interesting technical side issues not consistent with these goals have been omitted or relegated to boxes.

We have tried to be as sympathetic to our readers as possible, by providing ample interesting examples, exercises, and applications. When a computation in coordinates is easiest, we give it and don't try to hide things behind complicated invariant notation. On the other hand, index-free notation can often provide valuable geometric, and sometimes computational, insight so we have tried to simultaneously convey this flavor.

The only prerequisites required are a solid undergraduate course in linear algebra and a classical course in advanced calculus. At isolated points in the text some contacts are made with other subjects. For students, this provides a good way to link this material with other courses. These links do not require extra background material, but it is more meaningful when the bond is made. For example, Chapter 1 (and Appendix A) link with point-set topology, the boxes in Chapter 2 and Appendices B, C, D, are connected with functional analysis, Section 4.3 relates to ordinary differential equations, Chapter 3, Section 7.5, and Appendix E are linked to differential topology and algebraic topology, and finally Chapter 8 on applications is connected with applied mathematics, physics, and engineering.

This book is intended to be used in courses as well as for reference. The sections are, as far as possible, lesson sized, if the boxed material is omitted. For some sections, like 2.5, 4.2, or 7.5 two lecture hours are required. A standard course for mathematics graduate students, for example, could omit Chapter 1 and the boxes entirely and do Chapters 2 through 7 in one semester with the possible exception of Section 7.4. The instructor could then assign certain boxes or appendices for reading and choose among the applications of Chapter 8 according to taste. A shorter course, or a course for advanced undergraduates probably should omit all boxes, spend about two lectures on Chapter 1 for reviewing background point set topology, and cover Chapters 2 through 7 with the exception of Sections 4.4, 7.4, 7.5 and all the material relevant to volume elements induced by metrics, the Hodge star, and codifferential operators in Sections 6.2, 6.4, 6.5, and 7.2. A more applications oriented course could omit Chapter 1, review without proofs the material of Chapter 2 and cover Chapters 3 to 8 omitting the boxed materials and Sections 7.4 and 7.5. For such a course the instructor should keep in mind that while Sections 8.1 and 8.2 use only elementary material, Section 8.3 relies heavily on the Hodge star and codifferential operators, while Section 8.4 consists primarily of applications of Frobenius' theorem dealt with in Section 4.4. The appendices, included for completeness, contain technical proofs of facts used in isolated places in the text.

The notation in the book is as standard as conflicting usages in the literature allow. We have had to compromise among utility, clarity, clumsiness, and absolute precision. Some possible notations would have required too much interpretation on the part of the novice while others, while precise, would have been so dressed up in symbolic decorations that even an expert in the field would not recognize them. We have used boldface symbols to help the reader distinguish objects; for the most part, linear spaces, linear operators and abstract tensor fields are in boldface italics, while manifolds, points, point mappings, and tensor components are in lightface italics. Strict compliance is not always possible. This usage is only to help the reader distinguish symbols to which no further significance should be attributed.

In a subject as developed and extensive as this one, an accurate history and crediting of theorems is a monumental task, especially when so many results are folklore and reside in private notes. We have indicated some of the important credits where we know of them, but we did not undertake this task systematically. We hope our readers will inform us of these and other shortcomings of the book so that, if necessary, corrected printings will be possible.

The reference list at the back of the book is confined to works actually cited in the text. These works are cited by author and year like this: deRham [1955].

During the preparation of the book, much valuable advice was provided by Alan Weinstein. Our other teachers and collaborators from whom we learned the material and who inspired, directly and indirectly, various portions of the text are too numerous to mention individually. We hereby thank them all collectively. Finally, we thank Connie Calica for her careful typing of the manuscript.

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BACKGROUND NOTATION

The reader is assumed to be familiar with usual notations of set theory such as $\in$, $\cup$, $\cap$ and with the concept of a mapping. If A and B are sets and $f: A \to B$ is a mapping, we often write $a \mapsto f(a)$ for the effect of the mapping on the element $a \in A$; "iff" stands for "if and only if" (= "if" in definitions).

Other notations we shall use without explanation include the following:

$\mathbb{R}, \mathbb{C}$	real, complex numbers
$\mathbb{Z}, \mathbb{Q}$	integers, rational numbers
$A \times B$	Cartesian product
$\mathbb{R}^n, \mathbb{C}^n$	Euclidean n-space, complex n-space
$(x^1, \ldots, x^n) \in \mathbb{R}^n$	point in $\mathbb{R}^n$
$A \setminus B$	set theoretic difference
I or Id	identity map
$f^{-1}(B)$	inverse image of B under f
$\Gamma_f = \{(x, f(x)) \mid x \in$ domain of $f\}$	graph of f
$\inf A$	infinimum (greatest lower bound) of $A \subset \mathbb{R}$
$\sup A$	supremum (least upper bound) of $A \subset \mathbb{R}$
$e_1, \ldots, e_n$	basis of an n-dimensional vector space
$\ker T$, range T	kernel and range of a linear transformation T
▲	end of an example
■	end of a proof
▼	proof of a lemma is done, but the proof of the theorem goes on.

These modifications of the Halmos symbol ■ are notations of Alan Weinstein.

Contents

CHAPTER 1

Topology

The purpose of this chapter is to introduce just the right amount of topology for later requirements. It is assumed that the reader has had a course in advanced calculus and so is acquainted with open, closed, compact, and connected sets in Euclidean space (see for example Marsden [1974a] and Rudin [1976]). If this background is weak, the reader may find the pace of this chapter too fast. If the background is under control, the chapter should serve to collect, review, and solidify concepts in a more general context.

A key concept in manifold theory is that of a differentiable map between manifolds. However, manifolds are also topological spaces and differentiable maps are continuous. Topology is the study of continuity in a general context; it is therefore appropriate to begin with it.

Topology often involves interesting excursions into pathological spaces and exotic theorems. Such excursions are deliberately minimized here. The examples will be ones most relevant to later developments, and the main thrust will be to obtain a working knowledge of continuity, connectedness, and compactness.

1.1 TOPOLOGICAL SPACES

Abstracting our ideas about open sets in $\mathbb{R}^n$, we shall first consider the notion of a topological space.

1.1.1 Definition. *A **topological space** is a set S together with a collection $\mathcal{O}$ of subsets called **open sets** such that*

(T1) $\varnothing \in \mathcal{O}$ *and* $S \in \mathcal{O}$;

(T2) *If* $U_1, U_2 \in \mathcal{O}$, *then* $U_1 \cap U_2 \in \mathcal{O}$;

(T3) *The union of any collection of open sets is open.*

A basic example is the *real line*. We choose $S = \mathbb{R}$, with $\mathcal{O}$ consisting of all sets that are unions of open intervals. Thus, as exceptional cases, the empty set $\varnothing \in \mathcal{O}$ and $\mathbb{R}$ itself belong to $\mathcal{O}$. Thus (T1) holds. For (T2), let U_1 and $U_2 \in \mathcal{O}$; to show $U_1 \cap U_2 \in \mathcal{O}$, we can suppose $U_1 \cap U_2 \neq \varnothing$. If $x \in U_1 \cap U_2$ then x lies in an open interval $]a_1, b_1[\subset U_1$ and $x \in]a_2, b_2[\subset U_2$. Let $]a_1, b_1[\cap]a_2, b_2[=]a, b[$ (so $a = \max(a_1, a_2)$ and $b = \min(b_1, b_2)$). Thus $x \in]a, b[\subset U_1 \cap U_2$. Hence $U_1 \cap U_2$ is the union of such intervals, so is open. Finally, (T3) is clear by definition.

Similarly, $\mathbb{R}^n$ may be topologized by declaring a set to be open if it is a union of open rectangles. An argument similar to the one just given shows that this is a topology, called the *standard topology* on $\mathbb{R}^n$. Open intervals in $\mathbb{R}$ and open rectangles in $\mathbb{R}^n$ are each examples of a *basis* $\mathcal{B}$ for a topology; i.e., every open set is a union of sets in $\mathcal{B}$.

Any set S can be topologized in an obvious manner in two ways. The *trivial topology* on S consists of $\mathcal{O} = \{\varnothing, S\}$. The *discrete topology* on S is defined by $\mathcal{O} = \{A \mid A \subset S\}$; i.e., $\mathcal{O}$ consists of all subsets of S.

Topological spaces are specified by a pair $(S, \mathcal{O})$; we shall, however, just write S if there is no danger of confusion.

1.1.2 Definition. *Let S be a topological space. A set $A \subset S$ is called **closed** if its complement $S \setminus A$ is open. The collection of closed sets will be denoted $\mathcal{C}$.*

For example, the closed interval $[0, 1] \subset \mathbb{R}$ is closed as it is the complement of the open set $]-\infty, 0[\cup]1, \infty[$.

1.1.3 Proposition. *The closed sets in a topological space satisfy*:

(C1) $\varnothing \in \mathcal{C}$ *and* $S \in \mathcal{C}$;

(C2) *If* $A_1, A_2 \in \mathcal{C}$ *then* $A_1 \cup A_2 \in \mathcal{C}$;

(C3) *the intersection of any collection of closed sets is closed.*

Proof. (C1) follows from (T1) since $\varnothing = S \setminus S$, $S = S \setminus \varnothing$. The relations

$$S \setminus (A_1 \cup A_2) = (S \setminus A_1) \cup (S \setminus A_2)$$

and

$$S \setminus \left(\bigcap_{i \in I} B_i \right) = \bigcup_{i \in I} (S \setminus B_i)$$

for $\langle B_i \rangle_{i \in I}$ a family of closed sets show that (C2),(C3) are equivalent to (T2),(T3), respectively. ■

Closed rectangles in $\mathbb{R}^n$ are closed sets, as are closed balls, one-point sets, and spheres. Not every set is either open or closed. For example, the interval $[0, 1[$ is neither an open nor a closed set. In the discrete topology on S any set $A \subset S$ is both open and closed, whereas in the trivial topology any $A \neq \varnothing$ or S is neither.

Closed sets can be used to introduce a topology just as well as open ones. Thus, if $\mathcal{C}$ is a collection satisfying (C1)–(C3) and $\mathcal{O}$ consists of the complements of sets in $\mathcal{C}$, then $\mathcal{O}$ satisfies (T1)–(T3).

A considerable amount of topological terminology turns out to be useful. We shall introduce some now.

1.1.4 Definition. *An **open neighborhood** of a point u in a topological space S is an open set U such that $u \in U$. Similarly, for a subset A of S, U is an open neighborhood of A if U is open and $A \subset U$. A **neighborhood** of a point (or a subset) is a set containing some open neighborhood of the point (or subset).*

If $x \in \mathbb{R}$, neighborhoods of x are, for example, $]x-1, x+3]$, $]x-\varepsilon, x+\varepsilon[$ for any $\varepsilon > 0$, and $\mathbb{R}$ itself; only the last two are open neighborhoods. The set $[x, x+2[$ contains the point x but is not one of its neighborhoods.

In the trivial topology on a set S, there is only one neighborhood of any point, namely S itself. In the discrete topology any subset containing p is a neighborhood of the point $p \in S$, since $\{p\}$ is an open set.

1.1.5 Definition. *A topological space is called **first countable** if for each $u \in S$ there is a sequence $\langle U_1, U_2, \ldots \rangle = \langle U_n \rangle$ of neighborhoods of u such that for any neighborhood U of u, there is an n such that $U_n \subset U$. The topology is called **second countable** if it has a countable basis.*

Topological spaces of interest to us will largely be second countable. For example $\mathbb{R}^n$ is second countable since it has the countable basis formed by rectangles with rational side length, centered at points all of whose coordinates are rational. Clearly every second-countable space is also first countable, but the converse is false. For example if S is an infinite noncountable set, the discrete topology is not second countable, but S is first countable, since $\{p\}$ is a neighborhood of $p \in S$. The trivial topology on S is second countable (see Exercises 1.1I, 1.1J for more interesting counter-examples).

A basic fact about second-countable spaces is the following statement due to Lindelöf.

1.1.6 Proposition. *Every covering of a set A in a second-countable space S by a family of open sets U_α (that is $\cup_\alpha U_\alpha \supset A$) contains a countable subcollection $\{U_{\alpha_1}, U_{\alpha_2}, \ldots\}$ also covering A.*

Proof. Let $\mathcal{B} = \{B_n\}$ be a countable basis for the topology of S. For each $p \in A$ there are indices n and α such that $p \in B_n \subset U_\alpha$. Let $\mathcal{B}' = \{B_n | \text{there exists an } \alpha \text{ such that } B_n \subset U_\alpha\}$. Now let U_{α_n} be one of the U_α that includes the element B_n of $\mathcal{B}'$. Since $\mathcal{B}'$ is a covering of A, the countable collection $\{U_{\alpha_n}\}$ covers A. ■

The terminology of closure and interior is also very useful.

1.1.7 Definition. *Let S be a topological space and $A \subset S$. Then the **closure** of A, denoted $\mathrm{cl}(A)$ is the intersection of all closed sets containing A. The **interior** of A, denoted $\mathrm{int}(A)$ is the union of all open sets contained in A. The **boundary** of A, denoted $\mathrm{bd}(A)$ is defined by*

$$\mathrm{bd}(A) = \mathrm{cl}(A) \cap \mathrm{cl}(S \setminus A)$$

By (C3), $\mathrm{cl}(A)$ is closed and by (T3), $\mathrm{int}(A)$ is open. Note that as $\mathrm{bd}(A)$ is the intersection of closed sets, $\mathrm{bd}(A)$ is closed, and $\mathrm{bd}(A) = \mathrm{bd}(S \setminus A)$. Note that A is open iff $A = \mathrm{int}(A)$ and closed iff $A = \mathrm{cl}(A)$.

For example, on $\mathbb{R}$,

$$\mathrm{cl}([0,1[) = [0,1], \quad \mathrm{int}([0,1[) = (0,1) \quad \text{and} \quad \mathrm{bd}([0,1[) = \{0,1\}.$$

The reader is assumed to be familiar with examples of this type from advanced calculus.

Some notions building on these are as follows.

1.1.8 Definitions. *A subset A of S is called **dense** in S if $\mathrm{cl}(A) = S$ and is called **nowhere dense** if $S \setminus \mathrm{cl}(A)$ is dense in S.*

*S is called **separable** if it has a countable dense subset.*

*A point in S is called an **accumulation point** of the set A if each of its neighborhoods contains points of A other than itself. The set of accumulation points of A is called the **derived set** of A and is denoted by $\mathrm{der}(A)$.*

*A point of A is said to be **isolated** if it has a neighborhood in S containing no other point of A than itself.*

The set $A = [0,1[\cup \{2\}$ in $\mathbb{R}$ has 2 as its only isolated point, $\mathrm{int}(A) =]0,1[$, $\mathrm{cl}(A) = [0,1] \cup \{2\}$ and $\mathrm{der}(A) = [0,1]$. In the discrete topology on a set S, $\mathrm{int}\{p\} = \mathrm{cl}\{p\} = \{p\}$, for any $p \in S$.