3-MANIFOLDS

BY

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PREFACE

The aim of this work is to provide a consistent and systematic treatment of the topological structure of 3-dimensional manifolds. Our ultimate goal would be to provide (as have been done in dimension two) a "list" containing exactly one 3-manifold from each homeomorphism class together with an effective procedure for determining where a "given" 3-manifold belongs in this list. While this problem remains far from solution, the period since Papakyriakopoulos' proofs of Dehn's lemma and the loop and sphere theorems has produced considerable progress toward a solution and we have attempted to provide an organized account of these developments. We have excluded two topics: knot theory – which is a subject in itself and for which there are several reference works ([10], [16], [75]) and a consideration of local problems (wild embeddings, etc.) which are nicely covered in [13].

A basic principle in n-manifold topology is that k-dimensional homotopy theoretic information translates nicely to topological information provided the codimension, $n-k$, is sufficiently large $(n-k \geq 3)$. This, together with duality, allows one to concentrate problems into the middle dimension, $[n/2]$, provided that $n - [n/2] \geq 3$. Of course this condition fails for $n < 5$. Also for $n = 3$ the middle dimension, one, involves the fundamental group – the only nonabelian homotopy group. This may help explain why the techniques (and results) in 3-manifold theory differ from the general theory $(n \geq 5)$ and why the algebraic invariants involved are almost entirely group theoretic. The theme of this work is the role of the fundamental group of a 3-manifold in determining its topological structure.

We assume the reader is familiar with the basic elements of algebraic topology (covering spaces, Poincaré duality, the Hurewicz isomorphism theorem, and related topics). We will also make use of some facts from piecewise linear (p.l.) topology and from combinatorial group theory. In Chapter 1 we give a summary of the p.l. topology (regular neighborhoods, general position, etc.) which we use. We have included proofs of several theorems from group theory. Many of these, involving the structure of subgroups or quotient groups of a given group, translate (via covering spaces) to topological theorems and we have given topological proofs. We suggest that the reader may find it profitable to combine a study of 3-manifold topology with a study of combinatorial group theory.

We have placed exercises at appropriate points throughout the text. We have implicitly included many others by leaving details to be supplied by the reader.

Most of the material covered has appeared elsewhere in some form. We have made an effort to extend to nonorientable and/or bounded manifolds results which were previously known only for orientable and/or closed manifolds. We feel that we have achieved some economy in presentation by permuting the historical order of development. In particular, we have introduced the concept of "incompressible surface" as early as possible. Incompressible surfaces have turned out to be highly representative of the manifolds containing them. Combined with the tools provided by the loop and sphere theorems an analysis of the incompressible surfaces in a 3-manifold has proved to be the most effective approach to understanding the structure of the manifold. The most dramatic evidence of this is given in Chapter 13 where these ideas are used to show that a large class of 3-manifolds are completely determined by their fundamental group systems.

We have included a list of references which we hope will give proper credit to the original sources of the key ideas in the subject and will provide sufficient leads to further study. We have not attempted to provide a complete list of related works and apologize for all omissions.

This work developed through courses given at Rice University. I wish to express my gratitude to my students and colleagues for the stimulus they have given me.

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CHAPTER I
PRELIMINARIES

We will approach the study of 3-manifolds from the piecewise linear (p.l.) point of view. This choice is prompted partly by tradition, partly by the ease with which low dimensional polyhedra may be visualized, and partly because of the technical convenience afforded by the "finiteness" of polyhedra. On the last point, while many of the standard arguments can be translated from the p.l. theory to the differentiable theory by replacing general position by transversality, there are some notable exceptions, e.g. proofs of the loop theorem and the existence of hierarchies, which do not seem well suited to differentiable techniques. Since each topological 3-manifold has a p.l. structure unique up to p.l. homeomorphism [71], [6] and a differentiable structure unique up to diffeomorphism [73], [112], our choice causes no loss of generality – theorems in the p.l. setting have direct analogues in the differentiable setting (as well as in the locally flat topological setting). Our choice does prohibit consideration of "wild" (nonlocally flat) embeddings of submanifolds, and we do not consider such matters. Thus we work entirely within the p.l. category; from Chapter 2 on the prefix p.l. will be understood to be attached to the terms manifold, submanifold, map, etc., unless otherwise indicated.

We will assume some basic elements from algebraic topology (e.g. Poincaré duality, Hurewicz Theorem, etc.), from group theory (most of which can be found in [63]), and from p.l. topology. For completeness we state, in the remainder of this chapter, the facts from p.l. topology one needs to begin with, and refer to [29], [47], [88], or [114] for a systematic development of the theory.

Definitions

We will denote n-dimensional Euclidean space by R^n, the unit ball $\{x \in R^n : \|x\| \leq 1\}$ by B^n, and the unit sphere $\{x \in R^n : \|x\| = 1\}$ by S^{n-1} and will call a space homeomorphic to $B^n(S^{n-1})$ an *n-cell* (*(n−1)-sphere*).

A (topological) *n-manifold* is a separable metric space each of whose points has an open neighborhood homeomorphic to either R^n or to $R^n_+ = \{x \in R^n : x_n \geq 0\}$. The *boundary* of an n-manifold M, denoted ∂M, is the set of points of M having neighborhoods homeomorphic to R^n_+; the *interior* of M, denoted Int M, is $M - \partial M$. By invariance of domain ∂M is either empty or an (n−1)-manifold and $\partial\partial M = \emptyset$. A manifold is *closed* if it is compact and has empty boundary and is *open* if it has no compact component and has empty boundary.

We will view a *simplicial complex* as a locally finite collection, K, of (closed) simplexes in some R^n satisfying

(i) If $\sigma \in K$ and τ is a face of σ, then $\tau \in K$.

(ii) If $\sigma, \tau \in K$, then $\sigma \cap \tau$ is a face of both σ and of τ.

We will denote the underlying space of K by $|K| = \cup\{\sigma : \sigma \in K\}$. By a *subdivision* of K we mean a simplicial complex L such that $|L| = |K|$ (as sets) and each simplex of L lies in some simplex of K. For simplicial complexes K_1, K_2 a map $f : |K_1| \to |K_2|$ is *piecewise linear* provided there exist subdivisions L_1 of K_1 and L_2 of K_2 with respect to which f is simplicial i.e. f takes vertices of L_1 to vertices of L_2 and takes each simplex of L_1 linearly (in terms of barycentric coordinates) onto a simplex of L_2. It is an elementary, but not altogether trivial, fact that the composition of piecewise linear maps is piecewise linear.

A *triangulation* of a space X is a pair (T, h) where T is a simplicial complex and $h : |T| \to X$ is a homeomorphism. Two triangulations (T_1, h_1) and (T_2, h_2) are *compatible* provided $h_2^{-1}h_1 : |T_1| \to |T_2|$ is piecewise linear.

For K a simplicial complex and σ a simplex of K, the star of σ with respect to K, $st(\sigma, K)$, is the subcomplex of K consisting of all simplexes of K which meet σ together with all their faces, and the *link* of σ with respect to K, $1k(\sigma,K)$, is the subcomplex consisting of all simplexes of K which do not meet σ but which are faces of some simplex of K containing σ.

A triangulation (T,h) of an n-manifold, M, is *combinatorial* provided that for each vertex v of T, $|1k(v,T)|$ is piecewise linearly homeomorphic to an (n–1)-simplex or the boundary of an n-simplex according as $h(v) \in \partial M$ or $h(v) \in \text{Int } M$. This implies the more general fact that for each simplex σ of T, $|1k(\sigma,T)|$ is p.l. homeomorphic to an $(n-\dim\sigma-1)$-simplex or the boundary of an $(n-\dim\sigma)$-simplex according as $h(\sigma) \subset \partial M$ or $h(\sigma) \not\subset \partial M$. If (K,h) is a combinatorial triangulation of M and L is a subdivision of K then (L,h) is also a combinatorial triangulation of M (c.f. [2] or [77]). A *p.l. structure* on a manifold M is a maximal, non-empty collection of compatible combinatorial triangulations of M. By a *p.l. manifold* we will mean a manifold M together with a p.l. structure on M. A map $f: M_1 \to M_2$ between p.l. manifolds is a *p.l. map* provided that for some (hence any) triangulations (T_i,h_i) $i = 1,2$ of M_i in the associated p.l. structures $h_2^{-1}fh_1 : |T_1| \to |T_2|$ is piecewise linear. We note [55] that there exist manifolds with inequivalent p.l. structures (i.e. homeomorphic p.l. manifolds which are not p.l. homeomorphic) and there exist manifolds with no p.l. structure whatever. The possibility that such manifolds might still admit (non-combinatorial) triangulations is still open and has some interesting consequences (see [93]). As previously noted such difficulties do not arise in low dimensions: each manifold of dimension at most 3 has a p.l. structure unique up to p.l. homeomorphism.

A submanifold N of a p.l. manifold M is a *p.l. submanifold* if there is a triangulation (T,h) in the p.l. structure on M and a subcomplex S of T such that $(S,h||S|)$ is a combinatorial triangulation of N (and hence determines a p.l. structure on N). This definition allows local knotting (i.e. the pair $(|1k(v,T)|, |1k(v,S)|)$ need not be p.l. homeomorphic to the standard sphere or ball pair of appropriate dimensions); however, if dim $M \leq 3$, all p.l. submanifolds are locally unknotted. A submanifold N of M is *proper* in M if $N \cap \partial M = \partial N$.

By an *orientation* of a p.l. n-manifold, M, we will mean a consistent orientation of the n-simplexes in a triangulation T of M. Here an orientation of an n-simplex is an equivalence class, modulo even permutation, of orderings of its vertices. Using square brackets for equivalence classes and the convention $-[v_0, v_1, \cdots, v_n]$ to denote the opposite orientation (i.e. $[v_1, v_0, \cdots, v_n]$), the orientation on the (n−1)-face opposite v_i *induced* by the orientation $[v_0, \cdots, v_n]$ is defined to be $(-1)^i[v_0, \cdots, \hat{v}_i, \cdots, v_n]$. Thus an orientation of M is a choice of an orientation for each n-simplex of T such that if an (n−1)-simplex τ is a face of two n-simplexes σ_1 and σ_2 of T, then the orientation on τ induced from that on σ_1 is opposite to the one induced from σ_2. Clearly a compact, connected n-manifold is *orientable* if and only if $H_n(M, \partial M) = Z$ and an orientation of M corresponds to a choice of generator for Z. We use the terms *oriented manifold* to mean a manifold together with a choice of orientation for it, and *unoriented manifold* to mean one which has not been oriented (whether or not it is possible to do so), and *nonorientable manifold* to mean one which can't be oriented.

Basic Theorems

By a p.l. *n-cell* (p.l. (n−1)-*sphere*) we mean a p.l. manifold p.l. homeomorphic to an n-simplex (its boundary).

Many of the elementary theorems, including the following three can be found in the early works of J. W. Alexander [2], and M. H. A. Newman [76], [77] as well as the reference works mentioned earlier.

1.1. THEOREM. *If* M *is a p.l.* n-sphere *and* C *is a p.l. submanifold which is a p.l.* n-cell, *then* $\overline{M - C}$ *is a p.l. submanifold which is a p.l.* n-cell.

1.2. THEOREM. *If* C *is a p.l.* n-cell, *then any p.l. homeomorphism of* ∂C *to itself can be extended to a p.l. homeomorphism of* C *to itself.*

1.3. THEOREM. *If* M *is a p.l.* n-*manifold and* C *is a p.l.* n-*cell such that* $M \cap C = \partial M \cap \partial C$ *is a p.l.* (n−1)-*cell (as a p.l. submanifold of both* M *and of* C), *then* M *is p.l. homeomorphic to* $M \cup C$.

The next two theorems are due to V. K. A. M. Gugenheim [30].

1.4 THEOREM. *If* M *is a p.l.* n-*cell or a p.l.* n-*sphere, then any orientation preserving p.l. homeomorphism of* M *onto itself is p.l. isotopic to the identity.*

1.5 THEOREM. *If* M *is a p.l.* n-*manifold,* C_1 *and* C_2 *are p.l.* n-*cells (as p.l. submanifolds) in* Int M *and* X *is any closed subset of* M *such that* $C_1 \cup C_2$ *lies in a component of* $M - X$, *then there is a p.l. isotopy* $\phi : M \times I \to M$ *such that* $\phi_0 = 1$, $\phi_t | X = 1$ *for all* $t \in I$, *and* $\phi_1(C_1) = C_2$.

Regular Neighborhoods

The theory of regular neighborhoods was developed by J. H. C. Whitehead [109]. We describe the essential features.

If K is a simplicial complex, σ is a simplex of K and τ is a face of σ, with $\dim \tau = \dim \sigma - 1$, which is not a proper face of any other simplex of K, then the complex $K - \{\sigma, \tau\}$ is said to be obtained from K by an *elementary collapsing*. If a subcomplex L of K is obtained from K by a finite sequence of elementary collapsings, we say K collapses to L and denote this $K \searrow L$. Note that if $K \searrow L$, then $|L|$ is a strong deformation retract of $|K|$.

Suppose P is a compact *polyhedron* in a p.l. n-manifold M (i.e. P is the image of a finite subcomplex of some allowable triangulation of M). By a *regular neighborhood* of P in M we mean a p.l. n-submanifold N of M such that there is a triangulation (T,L) in the p.l. structure on M and finite subcomplexes K, L of T with $K \searrow L$, $h(|K|) = N$, and $h(|L|) = P$. Note that a regular neighborhood of P may or may not be a neighborhood of P in the traditional sense (i.e. P need not be in the topological interior of N).

1.6. THEOREM. *Let* M *be a p.l. manifold,* (T,h) *a triangulation in the p.l. structure on* M, *and* L *a finite subcomplex of* T. *Let* $N(L,T) = \bigcup_{\sigma \in L} st(\sigma,T)$. *Then* $h(|N(L,T)|)$ *is a regular neighborhood of* $h(|L|)$ *provided*

(i) *Each simplex of* T *which has all its vertices in* L *is in* L (*i.e.* T *is full in* L), *and*

(ii) *If* $\sigma \in N(L,T)$ *and* $\sigma \cap |L| = \emptyset$, *then* $lk(\sigma, N(L,T)) \cap L$ *collapses to a vertex.*

1.7. COROLLARY. $h(|N(L'',T'')|)$ *is a regular neighborhood of* $h(|L|)$; *where* T'' *denotes the second barycentric subdivision of* T.

1.8. THEOREM. *Let* M *be a p.l. manifold,* P *a compact polyhedron in* M *and* N_1 *and* N_2 *regular neighborhoods of* P *in* M, *then:*

(i) *There is a p.l. homeomorphism* $h: N_1 \to N_2$,

(ii) *If* $P \subset \text{Int } N_i (i=1,2)$, *we can require that* $h|P = 1$.

(iii) *If* $N_i \cap \partial M$ *is a regular neighborhood of* $P \cap \partial M (i=1,2)$ (*hence* $N_i \cap \partial M = \emptyset$ *if* $P \cap \partial M = \emptyset$), *there is a p.l. isotopy* $f: M \times I \to M$ *such that* $f_0 = 1$ *and* $f_1(N_1) = N_2$.

(iv) *If, in* (iii), $P \cap M - N_i = \emptyset \ (i=1,2)$, *then we can require that* $f_t|P = 1$ *for all* $t \in I$.

1.9. COROLLARY. *If* (T,h) *is an allowable triangulation of a p.l.* n-*manifold* M *and* L *is a subcomplex of* T *which collapses to a vertex, then any regular neighborhood of* $h(|L|)$ *in* M *is a p.l.* n-*cell.*

1.10. COROLLARY. *If* M *is a p.l.* n-*manifold, then any regular neighborhood of* ∂M *in* M *is p.l. homeomorphic to* $\partial M \times I$.

General Position

A fundamental fact of p.l. topology is that two polyhedra in a p.l. manifold may be moved slightly to be in "general position" in the sense that their intersection is as simple as possible or, more generally, that a map of a polyhedron into a manifold can be approximated by one with simple singularities (self intersections). If a map $f: |K| \to R^n$ embeds the 0-skeleton of K onto a maximally independent set of points and is affine on each simplex of K, then it is a matter of elementary linear algebra to analyze the singularities of f, and this situation serves as a model for general position. However, if we subdivide K in order to make f simplicial (as a map into some triangulation of R^n), it no longer satisfies the above condition – simplexes must be introduced where intersections occur. Since subdivision is a necessity in piecing together local "general position" approximations to yield a global one for a map into a manifold, one is faced with the problem of providing an invariant definition of general position which preserves as much as possible the properties of the above mentioned model. Most of the treatments of general position known to me resolve this problem by accepting a weak definition – usually involving only the dimension of the singular set and neglecting "transversality" of intersections and/or the behavior of the map at a "branch point." Our approach, admittedly cumbersome and inelegant, is to spell out the properties of general position which we will subsequently use. We limit the generality to that actually needed. For a simplicial complex K, a map $f: |K| \to R^n$ is called *affine* if f maps each simplex of K linearly, in terms of barycentric coordinates, into R^n. For a map $f: X \to Y$ we define the *singular set*, S(f), of f to be the closure of $\{x \in X : \#(f^{-1}(f(x))) > 1\}$. We decompose S(f) as a disjoint union, $S(f) = \bigcup_{i \geq 1} S_i(f)$, by $S_i(f) = \{x \in S(f) : \#(f^{-1}(f(x))) = i\}$. Putting $\Sigma_i(f) = f(S_i(f))$, we call the points of $\Sigma_1(f)$ *branch points*, $\Sigma_2(f)$ *double points*, $\Sigma_3(f)$ *triple points*, and so on. For $x \in |K|$, K a simplicial complex, we define the *local dimension* of K at x, locdim(K,x), to be the maximal dimension of the (closed) simplexes of K containing x. A point $x \in |K|$ is called a *regular point* of K, if there is an open neighborhood of x in |K| homeomorphic to either R^q or to R^q_+, $q = \text{locdim}(K,x)$. Regular points of the second type will be called *boundary points* of K.

1.11. **Definition**. For $k < n \leq 3$, and K a finite k-complex, a map $f: |K| \to R^n$ is in *general position with respect to* K provided:

(i) f is an affine embedding on each simplex of K,

(ii) $\dim S_1(f) \leq n-3$, K has local dimension $(n-1)$ at each point of $S_1(f)$, and $f|(|K| - S_1(f))$ is an immersion.

(iii) for $i \geq 2$, $\dim S_i(f) \leq ik - (i-1)n$ (hence $S_i(f) = \emptyset$ for $i > n$), furthermore for $y \in \Sigma_i(f)$, and $f^{-1}(y) = \{x_1, \cdots, x_i\}$ $\sum_{j=1}^{i} \operatorname{locdim}(K, x_j) \geq (i-1)n$.

(iv) for $i \geq 2$ $S_i(f)$ contains a nonregular point only in the case $n = 3$, $i = 2$. In this case $S_2(f)$ contains only finitely many nonregular points and for each such point x the other point, x_1, of $f^{-1}(f(x))$ is a regular, nonboundary point and K has local dimension 2 at x and at x_1.

(v) for $i \geq 2$ and $y \in \Sigma_i(f)$, $f^{-1}(y)$ contains at most one boundary point; this occurs only when $n = 3$ and K has local dimension 2 at each point of $f^{-1}(y)$.

(vi) for $i \geq 2$ and $y \in \Sigma_i(f)$ a point such that K is regular at each point x_j of $f^{-1}(y)$, f is *transverse at* y in the sense that there exist maximally independent hyperplanes $H_1, \cdots, H_i$ through 0 with $\dim H_j = \operatorname{locdim}(K, x_j)$, a neighborhood N of y in R^n and a p.l. embedding $h: N \to R^n$ with $h(y) = 0$, and with hf taking a neighborhood of x_j in K onto a neighborhood of 0 in H_j or H_j^+ according as x_j is not or is a boundary point of K.

1.12 **Lemma**. *Suppose* K *is a finite complex of dimension* $k < n \leq 3$, A, B, C *are subcomplexes of* K *with* $K = A \cup B \cup C$, $A \cup B$ *a full subcomplex of* K *and* $A \cap C = \emptyset$. *Then given any affine map* $g: |K| \to R^n$ *such that* $g||B|$ *is in general position with respect to* B *and given* $\varepsilon > 0$ *there exists an affine map* $f: |K| \to R^n$ *satisfying*:

(a) $d(f(x), g(x)) < \varepsilon$ *for all* $x \in |K|$,

(b) $f||A \cup B| = g||A \cup B|$,

(c) $f||B \cup C|$ *is in general position with respect to* $B \cup C$, *and*

(d) *for each subcomplex* L *of* K *such that* $g||L|$ *is an embedding*, $f||L|$ *is also an embedding*.