THE CLASSICAL ELECTROMAGNETIC FIELD

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PREFACE

This book is intended as a text for a graduate course in electromagnetic theory. I began to write it after teaching such a course at M.I.T. and Northeastern University for several years and being dissatisfied with the existing texts. The reader will judge whether or not I have succeeded in writing a better one, but here I would like to show, at least obliquely, the reasons for my dissatisfaction, by outlining the text and my reasons for writing it as I have done.

The book is divided into two main parts: static fields and time-varying fields. Of course, this division is no novelty; it reflects history and is incorporated in large numbers of textbooks. It has the consequence that the electric and magnetic fields are considered as separate entities until they are amalgamated by special relativity and the Lorentz transformation. Since relativity theory tends these days to be taught earlier and earlier in the physics students' career, the possibility is now open of altering this canonical (static/time-varying) division. After the electrostatic field is discussed, one can, with some confidence, introduce the magnetic field as a result of viewing an electric field from a moving frame. This approach has some merit, and I have not dismissed it lightly. It is more basic, in a way, in that only one field need be postulated; the other can then be derived from it using relativistic transformations which must eventually be introduced anyway. And the symmetry between electric and magnetic fields that is made manifest by relativity is aesthetically very satisfying. But the amount of relativity theory that must be introduced before the magnetic field emerges from the transformed electric field is considerable. I think the danger of the student not seeing the wood for the trees outweighs the advantages of this procedure and so I have not adopted it.

When dealing with static fields, the field concept is, in fact, superfluous. All of electrostatics is comprised (in principle) in Coulomb's law for the force between two charges and in the law of superposition, and all of magnetostatics in Ampère's law for the force between two currents. The split of Coulomb's law into a field produced by one charge, which field then acts on another, is a convenience for static charges, but it is not conceptually necessary. A similar remark applies to the division of Ampère's law into the production of a field by one stationary current and its action on a second. It is only for time-varying fields that the field concept assumes its real importance as a way of preserving the conservation laws of energy and momentum. But since one is forced eventually to introduce the idea of field, it is useful pedagogically to introduce it as soon as possible, i.e., in connection with statics, to promote familiarity and ease with it. Nonetheless, I have also felt it important to clarify the status of the field concept, and so have prefaced the chapters on static fields by one called "Concepts of a Field Theory" which essentially contains an elaboration of the remarks above.

Beyond the static/time-varying division, there is a kind of threefold symmetry. The book treats essentially three kinds of fields: the electric, the magnetic, and their amalgam, the time-varying electromagnetic field. Each of the treatments follows the same pattern, and the shape of this pattern is partly given by the idea of the "summation problem."* By this is meant the problem of evaluating the field from an integral expression over the given, *known* sources, i.e., charges or currents. The summation problem is meant to stand in contrast to boundary-value problems with matter in which matter effectively acts as the source of charges or current that are *not* known, or given, in advance of solving the problem. I have found this division, into a definition of a field and its corresponding summation problem, a sound one pedagogically and am convinced of its usefulness.

Chapter by chapter, the book develops as follows. After the first chapter on concepts of a field theory Chapter 2, on electrostatics, discusses properties of the electric field and of the scalar potential Φ. The chapter ends with the superposition integral for the potential Φ due to an arbitrary continuous or discrete charge distribution.

The problem of actually evaluating this integral for charge distributions of one kind or another is then the summation problem for electrostatics, the subject of Chapter 3. In it are considered distributions which occupy a finite volume and the corresponding multipole expansion; two-dimensional and one-dimensional distributions; surface charges and double layers; and dipolar distributions. One merit of a rather complete discussion of the kinds of distributions possible is that many of them are forced to our attention later. In Green's theorem, for example, the concept of surface charge and double layer enter, but it is very useful to have encountered these before coping with whatever other difficulties Green's theorem may entail. Similarly, the discussion in this chapter of the external field of dipole distributions is immediately applicable to a later theory of dielectrics.

Chapter 4 is on boundary-value problems with perfect conductors. It might have been subtitled "The Field of Unknown Distributions" to emphasize the connection of this problem with the summation problem of surface charges which is that of the field of *known* distributions. In a boundary-value problem, the final state of electrical equilibrium corresponds to *some* surface distribution on the conductors, but one that is unknown *a priori*. The various methods that apply to the summation problem must apply here as well, in a suitably modified way. Thus, for a given geometry, there occur the same solutions of Laplace's equation, the same symmetry considerations, the same far field expansions, etc. Among the special

* The phrase "summation problem" is not mine; it is due to Sommerfeld. (A. Sommerfeld, *Electrodynamics*, Academic Press, N.Y., 1952, p. 38.)

topics of this chapter is, of course, the method of images, which I discuss rather more cursorily than in many other books. On the other hand, I have pointed out what is often neglected, that the basic idea of the method (which is really nothing but inspired guesswork) applies to homogeneous as well as inhomogeneous problems. The classical method of superposition of separated solutions of Laplace's equation is, of course, also discussed. There is a section on the use of integral equations which I have found to be pedagogically rewarding: the physics involved in setting up these equations is enlightening, and they are almost the only practical way of solving problems for other than the simplest geometrical shapes. It usually comes as a relief to the student to be able to treat *some* other boundary than the sphere or cylinder. These equations must usually be solved numerically, of course, but they are well adapted to computer solution. At the worst, some of them can be crudely solved by hand and even this is worth the effort (once). In a further effort to escape the tyranny of the sphere and the cylinder, I discuss *composite problems.* These are problems in which the geometry is, so to speak, made up of a sum of separable parts and, typically, they are problems involving more than one sphere, a sphere and a cylinder, a cylinder and a half-plane, etc.

Chapter 5 treats the general theory of boundary-value problems involving Laplace's equation, i.e., the Dirichlet and Neumann problems. This entails, of course, a discussion of Green's theorem which I have deliberately not introduced until this time, although there is logic enough in discussing it earlier. But I have found that although readers follow the mathematical derivation of Green's theorem easily enough, they are not so quick to understand its real nature, and when it is useful. In particular, if it is introduced *before* the discussion of boundary-value problems with conductors, they somehow feel that Green's theorem should be useful in solving them, which it is not.

Chapter 6 treats dielectrics, and here I break with the usual textbook treatments which I think poorly of, in general. Where they are not vague, they tend to be unsound. One exception to this is in the book by Purcell (B); anyone who is familiar with it will see resemblances here to his discussion and may infer correctly that I am indebted to it in several ways. Not the least of these ways has been the reassurance that someone else thinks poorly of many of the standardized treatments.

The problem of dielectrics is, in effect, the problem of calculating average internal fields of dipole distributions. It is, in large part, a problem in statistical mechanics but there is no satisfactory general solution that derives the macroscopic properties from space and time averages of the microscopic ones. Any theory must then be essentially postulatory. The postulate I make involves the field $\boldsymbol{E}_m$ of the equivalent surface and volume charges that correctly yield the *external* field of dielectric matter; it relates $\boldsymbol{E}_m$ to the mean polarization and applied field. Although the postulate is labeled explicitly as such, I have tried to make clear the rationale for it and have adduced experimental evidence about dielectric-filled capacitors to support it. Finally, and perhaps most importantly, I show that the discussion of dielectrics that is based on this hypothesis is precisely equivalent to the usual formalism involving the vector $\boldsymbol{D}$.

This work on dielectrics temporarily closes out electrostatics. The next natural subjects are perhaps force and energy in the electrostatic field. I have deferred these, however, until magnetostatics has been discussed, and have then discussed force and energy side by side for both electric and magnetic fields. In this way, hopefully, one learns both from the similarities and from the differences.

Chapters 7, 8, and 9 comprise a discussion of the magnetostatic field along lines which parallel, insofar as possible, those for the electrostatic field. Thus, in Chapter 7, Ampère's law for the force between currents is stated, the split is made into a field $\boldsymbol{B}$ produced by one current which then acts on the other, and from the definition of $\boldsymbol{B}$ so obtained, its divergence and curl are calculated. I have found it sounder pedagogically to proceed in this way, i.e., to calculate the properties of the field *directly* from its definition rather than to do this by first introducing auxiliary potentials.

Chapter 8 comprises the second summation problem, that of calculating the field of an arbitrary stationary current distribution. Although many books use the vector potential as the major tool for this problem, the magnetic scalar one is almost always superior in practice, and I have treated it in some detail. The vector potential comes into its own only for time-varying fields. There are few cases in magnetostatics where it is easier, or even as easy, to calculate the three components of the vector potential than it is to do the single integration that yields the scalar potential.

After the summation problem for currents, the parallel with the electrostatic field breaks down. There is no magnetic analog of the electrostatic boundary-value problems with conductors since there are, of course, no conductors of magnetism. But there are magnetic materials which are the analogs of dielectrics, and with analogous problems, and these are the subject of Chapter 9. Once again, one must cope with violently fluctuating internal fields and here I have followed the natural course of patterning the discussion on that of dielectrics. An average internal macroscopic field $\boldsymbol{B}_m$ is defined by equivalent surface currents and a postulate is made that relates the mean magnetization $\boldsymbol{M}$ to $\boldsymbol{B}_m$ and to the applied field. It is then shown that the postulational approach is equivalent to the usual theory involving the vector $\boldsymbol{H}$.

The last chapter in statics is entitled "Force and Energy in Static Fields." I have chosen to treat the electrostatic and magnetostatic cases in this one chapter, to learn from their similarities *and* dissimilarities. One disadvantage is that one must, in setting up the expression for magnetic field energy, simply quote some results that will be derived later from Faraday's induction law. This seems, however, a small price to pay for the economy of treatment that results by treating the electric and magnetic case together.

The second part of this book, starting with Chapter 11, treats the time-varying electromagnetic field. The pattern of the discussion follows those for the static electric and static magnetic fields. First, the differential equations of the fields—in this case the Maxwell equations—are presented. The fields are then related to the retarded potentials, which is to say, integral expressions over the currents. The summation problem for these potentials is discussed and only then are boundary-value problems considered. Finally, the most difficult subject, that of fields inside matter, is treated in general, and dielectrics are discussed in particular.

Chapter by chapter, then, Chapter 11 states Maxwell's equations, following the introduction of the displacement current as the postulate that it really is. The displacement current is sometimes derived by requiring that the continuity equation be satisfied between the plates of a condenser, but this derivation merely camouflages the essentially postulatory nature of the current. The chapter is quite conventional in the main, except that I have spent more than the usual time on the question of conservation laws.

Chapter 12 treats the relation of the special theory of relativity to electrodynamics. In an effort to stay within reasonable bounds and yet keep everything of interest, I have foregone a common approach which starts from the historical experiments (Michelson–Morley, etc.), and have preferred to spend the time and space gained to discuss concepts at the base of the theory, such as absolute time, inertial frames, the nature of a vector, etc., which are perhaps not examined in enough detail in general.

Chapter 13 on time-harmonic currents is the third summation problem in the book and, as such, it can lean to a degree on the previous two, although it is, of course, much more complicated in detail.

Chapter 14 on the fields of point charges in motion is, in effect, also a summation problem. I had originally thought of calling this chapter the "Summation Problem for Point Currents" to emphasize this fact. But the point nature of the current makes this kind of summation problem different enough from the previous ones that such a title is perhaps somewhat labored. I have, however, tried to emphasize the similarity of the approximations necessary to evaluate the fields of point currents to those for the time-harmonic case. The explicit form of the Lienard–Wiechert fields is perhaps somewhat deceiving. They do not constitute the answer to a problem but are, in effect, the problem itself since, in any particular case, approximations must be derived for the retarded time in terms of the present time. These approximations, low velocity, multipole, etc., are essentially the same as those made for the time-harmonic case.

Chapter 15 is on time-harmonic boundary-value problems with perfect conductors. I have chosen to study this idealized case first before discussing the physics of imperfect conductors. There are enough new concepts even in the idealized case —modes, guide wavelengths, cutoff frequencies, etc.—that still other concepts of skin depth and field penetration are best deferred. Important time-harmonic boundary problems are those of diffraction. I have tried to emphasize the approximate nature of the usual diffraction theory of physical optics and at least outline one rigorous solution of a diffraction problem; this is the problem of diffraction by a perfectly conducting half-plane first solved by Sommerfeld but presented here in a version due to Clemmow. Too frequently, the Kirchhoff theory, i.e., the application of Green's theorem to the Helmholtz equation, leads to formulae that are represented as solutions of the diffraction problem or, at least, whose approximate nature is not enough emphasized. The solution of diffraction problems, then, appears to be an exercise in the evaluation of Fresnel or Fraunhofer integrals. This is, of course, not the case.

Chapter 16 on fields in matter was difficult to write. The question of the behavior of time-varying fields inside matter has, of course, all the difficulties of the two static cases and some of its own as well. It is usually presented for dielectrics in terms of the unsatisfactory formalism involving $\boldsymbol{D}$ and the time-dependent polarization vector, $\boldsymbol{P}$. Having shown how these are superfluous for electrostatics, I have tried to do the same for time-dependent fields in matter. I have, moreover, tried to treat all kinds of matter on the same conceptual basis. The common denominator has been to consider that matter of whatever kind is an ensemble of currents: damped currents for conductors; localized polarization currents for dielectrics, etc. The one postulate that is common to almost all classical models of matter is then a generalized Ohm's law: at a point the current is proportional to the field with a proportionality constant Γ which may be complex. The consequences of this postulate are reasonable for conductors. But in dielectrics, the currents are localized at atomic sites whereas the field is ubiquitous so the postulated relation can hold only in some average sense. One is led to basic difficulties involving the difference between the average field in a volume and the effective field acting on a dipole. These are, of course, the same problems that face one in the theory of the static dielectric constant. I have tried to raise these problems to the surface and, without solving them, to give, by means of various one-dimensional and other models, some idea of the physics that is involved in them.

With the properties of matter elucidated, the last chapter discusses boundary value and other problems associated with fields in matter, including reflection at a dielectric interface and surface waves. I began to write a section on Cerenkov radiation for this chapter. Most of its characteristic features can be derived by using the previously shown fact that time-harmonic waves propagate in matter with a modified wave number. But actually to calculate the intensity of Cerenkov radiation, one must also have an expression for the screening effect of the medium. This effect is sometimes expressed by saying that the medium is like free space except that the velocity c is replaced by c/n, where n is the dielectric constant, and a charge c is replaced by the effective value e/n. But the only way known to me of calculating this last result is by using the formal Maxwell equations in terms of $\boldsymbol{D}$ and $\boldsymbol{H}$ and the formal constitutive relations. As the last paragraph suggests, I am not convinced of the accuracy of this standard formal theory and *am* convinced of its ambiguity. Moreover, having done without $\boldsymbol{D}$ and $\boldsymbol{H}$ throughout the rest of the book, I thought it just as well to do without them altogether, at the cost of having to ask the reader to look up the Cerenkov effect elsewhere.

A word about the references. They are meant to supplement the text, but also to be randomly stimulating. I have tried to include some recent journal articles even in the earlier chapters, not with the intent of reviewing the literature completely, but rather of showing by a few essentially arbitrary references that even the older parts of electromagnetic theory are far from closed books to research.

I am indebted to many people for help of one kind or another in the making of this book. I am especially grateful to Dr. John Jasperse, who taught a course based on preliminary notes for the book, and whose advice helped shape large outlines and clarify small details. I have also profited from the comments of Dr. Ronald Newburgh on Chapter 12, and from the errors brought to my attention by Mr. Carl Holmstrom, who read through the whole manuscript. Finally, I owe a considerable debt to Mrs. Connie Friedman and Mrs. Donna Dickinson, whose technical typing skill and ability at deciphering were invaluable in producing the typescript of the bulk of the book.

Despite this help, errors and obscurities undoubtedly remain. I would be made grateful, if not happy, by anyone who brings them to my attention at the address below.

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PREFATORY NOTES

In this book, references are cited in three different ways. First, the bibliography at the end of the book lists references which are of interest for more than one chapter; these are referred to in the text by citing the author's name, followed by a B (for bibliography) in parentheses. For example: Whittaker (B). Second, the references at the end of a chapter, that are primarily of interest in the chapter itself, are cited by the author's name plus an R in parentheses: Kennedy (R). Finally, there are ordinary footnotes indicated by standard footnote symbols.

The units used in this book are cgs (Gaussian).

CONTENTS

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CONCEPTS OF A FIELD THEORY

This is a book on the *electromagnetic* field, one of the many examples of fields that are important in physics and natural science. In the words of Morse and Feshbach (B), "Practically all of modern physics deals with fields: potential fields, probability fields, electromagnetic fields, tensor fields, and spinor fields." Since the concept of field is applied so widely, it is perhaps natural that it has taken on somewhat different contexts with, of course, an underlying common denominator. In this first brief chapter, we shall try to analyze the concept and its variants and so highlight the essential aspects of the electromagnetic field.

One mathematical common denominator is easy to isolate. Mathematically, a field is a function, or a set of functions considered as an entity, of the coordinates of a point in space (and possibly of time). For example, if the temperature is defined at every point in some volume, we say there is a *scalar temperature field* throughout that volume. If, in a moving fluid, the three vector components of velocity are known as a function of position in the fluid, they constitute as a whole a *vector velocity field.* In the theory of elasticity, the relative vector *displacements* of points of an elastic solid from their unstrained positions are described in terms of a double vectorial or, as it is more usually called, a *tensor field.* In modern physics, the Schrödinger probability amplitude ψ or the generalized Dirac spinor amplitudes are examples of fields.

It is worth noting that in this above list, there are really two kinds of fields. The first kind, exemplified by the temperature or velocity fields, is an idealization that is really defined only in a certain approximation of coarseness or fineness. For example, the velocity field of hydrodynamics is meaningful only in an average or continuum approximation in which the atomic and grainy structure of the fluid is not considered. This point is discussed in some detail in Morse and Feshbach (B) and we shall not elaborate on it here. By contrast, the Schrödinger or Dirac fields are not approximations to an underlying discontinuous physical model but must be assumed to exist no matter how finely space is divided.

To describe the evolution of the electromagnetic field concept, we recall some history that begins with Newton (1642–1726). One of the great laws of physics is Newton's universal law of gravitation. This law embodies the concept of *action at a distance*, according to which gravitational masses exert the forces they do on each other by virtue of their positions in space, the *intervening* space playing no active role. This is meant to contrast with forces which work via *contiguous action* whereby two masses at a distance exert forces which are transmitted by the intervening medium. For example, if several billiard balls are in contact in a row on a table and the first one is struck, the last one will move. Thus, the one billiard ball exerts a force on the other distant one, but by a mechanism which involves successive actions of the intermediate balls, the one moving the next, moving the next, etc. The concept of contiguous action is then quite different from that of action at a distance where no intermediate mechanism or medium is considered.

The work of Newton is relevant in a second way. His laws of the motion of point particles and rigid bodies paved the way for the development of the continuum mechanics of fluids and later of elastic bodies. Some tentative beginnings on the subject of fluid flow were made by Newton himself, but the real groundwork was later laid by John Bernoulli (1667–1748) and Euler (1707–1783). They bypassed the problem of the actual microscopic structure of fluids by adopting a *continuum* model and then applied Newton's laws of point mechanics to small elements of the continuum. The same idea was later applied to elastic solids, and the vibrations of these solids was discussed by applying Newton's laws to a small element of the solid, assuming that the forces acting on it were the stresses due to the rest of the solid, plus any external forces. Hydrodynamics was therefore formulated in terms of the velocity and acceleration of the moving fluid at every point, i.e., in terms of *velocity and acceleration vector fields.* The theory of elastic solids was similarly formulated in terms of *stress and strain tensorial fields.*

So much for mechanics; we turn now to electromagnetism. A basic law of electrostatics is Coulomb's law for the force between two charged particles. Except for the fact that the electric force can be either attractive or repulsive, whereas the gravitational force is always attractive, this law is obviously similar to Newton's law of gravitation. It was then considered from the time of its discovery as an example of *action at a distance*, in which two charges act on each other in a way that has nothing to do with the intervening medium. But this view began to be questioned, at least in the mind of Faraday (1791–1867), by his work on dielectric polarization. This phenomenon led him to attribute more and more importance to the intervening medium. We cannot go into the details of Faraday's results and reasoning but, for illustration, shall concentrate on one of his findings. This has to do with the effect of insulators or *dielectrics* on the capacitance of condensers. Consider a parallel-plate capacitor with air between its plates; it has a certain capacitance. If the air is replaced by a dielectric medium, the capacitance will be increased. Faraday viewed the phenomenon of enhanced capacitance as somehow due to the fact that the electric force generated by the charges on the plates was *weakened* by the dielectric medium. But *if changing the medium* that intervened between the charges *changed the force*, then somehow the forces must depend on, or be *transmitted by*, the medium. As a corollary of this view, Faraday considered that the essential feature of the interaction between charged particles was the lines of force that carried the "stresses" of the medium from one charge to another. These lines of force that extend from charge to charge through the medium were considered