Tensor Analysis and Continuum Mechanics

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Preface

Through several centuries there has been a lively interaction between mathematics and mechanics. On the one side, mechanics has used mathematics to formulate the basic laws and to apply them to a host of problems that call for the quantitative prediction of the consequences of some action. On the other side, the needs of mechanics have stimulated the development of mathematical concepts. Differential calculus grew out of the needs of Newtonian dynamics; vector algebra was developed as a means to describe force systems; vector analysis, to study velocity fields and force fields; and the calculus of variations has evolved from the energy principles of mechanics.

In recent times the theory of tensors has attracted the attention of the mechanics people. Its very name indicates its origin in the theory of elasticity. For a long time little use has been made of it in this area, but in the last decade its usefulness in the mechanics of continuous media has been widely recognized. While the undergraduate textbook literature in this country was becoming "vectorized" (lagging almost half a century behind the development in Europe), books dealing with various aspects of continuum mechanics took to tensors like fish to water. Since many authors were not sure whether their readers were sufficiently familiar with tensors, they either added a chapter on tensors or wrote a separate book on the subject. Tensor analysis has undergone notable changes in this process, especially in notations and nomenclature, but also in a shift of emphasis and in the establishment of a cross connection to the Gibbs type of vector analysis (the "boldface vectors").

Many of the recent books on continuum mechanics are only "tensorized" to the extent that they use cartesian tensor notation as a convenient shorthand for writing equations. This is a rather harmless use of tensors. The general, noncartesian tensor is a much sharper thinking tool and, like other sharp tools, can be very beneficial and very dangerous, depending on how it is used. Much nonsense can be hidden behind a cloud of tensor symbols and much light can be shed upon a difficult subject. The more thoroughly the new generation of engineers learns to understand and to use tensors, the more useful they will be.

This book has been written with the intent to promote such understanding. It has grown out of a graduate course that teaches tensor analysis against the background of its application in mechanics. As soon as each mathematical concept has been developed, it is interpreted in mechanical terms and its use in continuum mechanics is shown. Thus, chapters on mathematics and on mechanics alternate, and it is hoped that this will bring lofty theory down to earth and help the engineer to understand the creations of abstract thinking in terms of familiar objects.

Mastery of a mathematical tool cannot be acquired by just reading about it—it needs practice. In order that the reader may get started on his way to practice, problems have been attached to most chapters. The reader is encouraged to solve them and then to proceed further, and to apply what he has learned to his own problems. This is what the author did when, several decades ago, he was first confronted with the need of penetrating the thicket of tensor books of that era.

The author wishes to express his thanks to Dr. William Prager for critically reading the manuscript, and to Dr. Tsuneyoshi Nakamura, who persuaded him to give a series of lectures at Kyoto University. The preparation of these lectures on general shell theory gave the final push toward starting work on this book.

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Contents

CHAPTER 1

Vectors and Tensors

IT IS ASSUMED THAT the reader is familiar with the representation of vectors by arrows, with their addition and their resolution into components, i.e. with the vector parallelogram and its extension to three dimensions. We also assume familiarity with the dot product and later (p. 36) with the cross product. Vectors subjected to this special kind of algebra will be called Gibbs type vectors and will be denoted by boldface letters.

In this and the following sections the reader will learn a completely different means of describing the same physical quantities, called tensor algebra. Each of the two competing formulations has its advantages and its drawbacks. The Gibbs form of vector algebra is independent of a coordinate system, appeals strongly to visualization and leads easily into graphical methods, while tensor algebra is tied to coordinates, is abstract and very formal. This puts the tensor formulation of physical problems at a clear disadvantage as long as one deals with simple objects, but makes it a powerful tool in situations too complicated to permit visualization. The Gibbs formalism can be extended to physical quantities more complicated than a vector (moments of inertia, stress, strain), but this extension is rather cumbersome and rarely used. On the other hand, in tensor algebra the vector appears as a special case of a more general concept, which includes stress and inertia tensors but is easily extended beyond them.

1.1. Dot Product, Vector Components

In a cartesian coordinate system x, y, z (Figure 1.1) we define a reference frame of unit vectors $\mathbf{i}_x$, $\mathbf{i}_y$, $\mathbf{i}_z$ along the coordinate axes and with their help a force vector

$$\mathbf{P} = P_x\mathbf{i}_x + P_y\mathbf{i}_y + P_z\mathbf{i}_z \tag{1.1a}$$

and a displacement vector

$$\mathbf{u} = u_x\mathbf{i}_x + u_y\mathbf{i}_y + u_z\mathbf{i}_z. \tag{1.1b}$$

These formulas include the well-known definition of the addition of vectors by the parallelogram rule.

In mechanics the work W done by the force $\mathbf{P}$ during a displacement $\mathbf{u}$ is defined as the product of the absolute values P and u of the two vectors and of the cosine of the angle β between them:

$$W = Pu \cos \beta.$$

This may be interpreted as the product of the force and the projection of $\mathbf{u}$ on the direction of $\mathbf{P}$ or as the product of the displacement and the projection of the force on $\mathbf{u}$. It is commonly written as the dot product of the two vectors:

$$W = \mathbf{P} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{P} = Pu \cos \beta. \tag{1.2}$$

This equation represents the definition of the dot product and may be applied to any two vectors. Since the projection of a vector $\mathbf{u} = \mathbf{v} + \mathbf{w}$ on the direction of $\mathbf{P}$ is equal to the sum of the projections of $\mathbf{v}$ and $\mathbf{w}$, it is evident that the dot product has the distributive property:

$$\mathbf{P} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{P} \cdot \mathbf{v} + \mathbf{P} \cdot \mathbf{w}.$$

When any one of the unit vectors $\mathbf{i}_x$, $\mathbf{i}_y$, $\mathbf{i}_z$ is dot-multiplied with itself, the angle β of (1.2) is zero, hence

$$\mathbf{i}_x \cdot \mathbf{i}_x = \mathbf{i}_y \cdot \mathbf{i}_y = \mathbf{i}_z \cdot \mathbf{i}_z = 1.$$

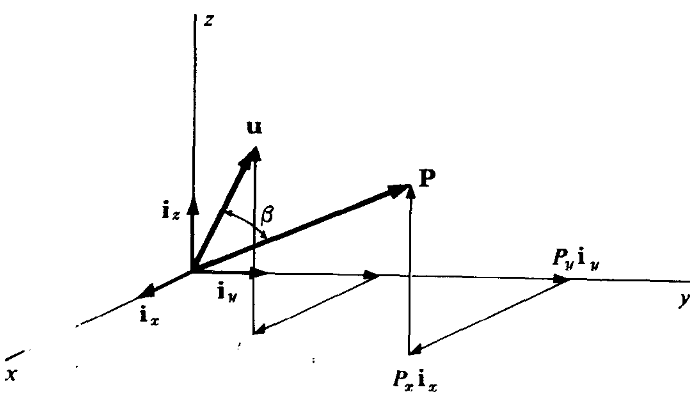

FIGURE 1.1 *Vectors in cartesian coordinates.*

If, on the other hand, two different unit vectors are multiplied with each other, they are at right angles and $\cos\beta = 0$, hence

$$\mathbf{i}_x \cdot \mathbf{i}_y = \mathbf{i}_y \cdot \mathbf{i}_z = \mathbf{i}_z \cdot \mathbf{i}_x = 0.$$

These relations may be combined into a single one:

$$\mathbf{i}_m \cdot \mathbf{i}_n = \delta_{mn}, \qquad m, n = x, y, z \tag{1.3}$$

if we introduce the *Kronecker delta*, δ_{mn}, by the equations

$$\begin{aligned} \delta_{mn} &= 1 \quad \text{if} \quad m = n, \\ \delta_{mn} &= 0 \quad \text{if} \quad m \neq n. \end{aligned} \tag{1.4}$$

We write the dot product of the right-hand sides of (1.1a, b):

$$\mathbf{P} \cdot \mathbf{u} = (P_x \mathbf{i}_x + P_y \mathbf{i}_y + P_z \mathbf{i}_z) \cdot (u_x \mathbf{i}_x + u_y \mathbf{i}_y + u_z \mathbf{i}_z).$$

When we multiply the two sums term by term, we encounter all the possible combinations of m and n in (1.3). Because of (1.4), only three of the nine products survive and we have

$$\mathbf{P} \cdot \mathbf{u} = P_x u_x + P_y u_y + P_z u_z, \tag{1.5}$$

a well-known formula of elementary vector algebra.

We try now to repeat this line of thought in a skew coordinate system. To simplify the demonstration, we restrict ourselves to two dimensions (Figure 1.2). We write the work, i.e. the dot product, first as the work done by $P_x \mathbf{i}_x$ plus that done by $P_y \mathbf{i}_y$:

$$\mathbf{P} \cdot \mathbf{u} = W = P_x(u_x + u_y \cos\alpha) + P_y(u_y + u_x \cos\alpha) \tag{1.6a}$$

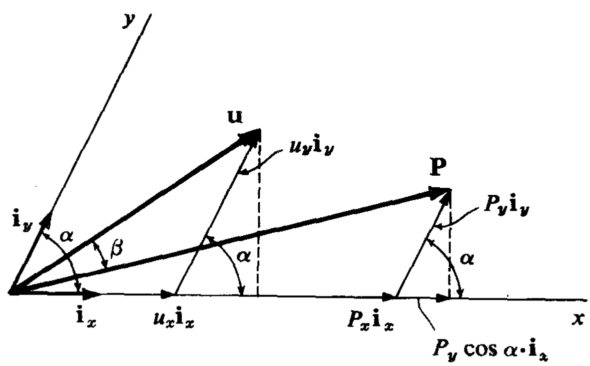

FIGURE 1.2 *Vectors in a skew rectilinear coordinate system.*

and then as the work done by **P** if the displacements $u_x\mathbf{i}_x$ and $u_y\mathbf{i}_y$ occur subsequently:

$$\mathbf{P}\cdot\mathbf{u} = W = u_x(P_x + P_y\cos\alpha) + u_y(P_y + P_x\cos\alpha). \tag{1.6b}$$

Each of these equations can be brought into the form

$$\mathbf{P}\cdot\mathbf{u} = P_xu_x + P_yu_y + (P_xu_y + P_yu_x)\cos\alpha.$$

This has yielded a result, but it has not increased our insight. It is better to leave equations (1.6) as they stand and to realize that we must deal with two different sets of components of each vector: (i) the usual ones, like P_x, P_y, which are obtained when **P** is made the diagonal of a parallelogram whose sides are parallel to the coordinate axes, and (ii) the components $(P_x + P_y\cos\alpha)$, $(P_y + P_x\cos\alpha)$, which are the normal projections of **P** on the axes x and y.

Before we embark upon a closer inspection of these components we introduce the notation which is fundamental for tensor theory and which will be used from now on in this book. Instead of components P_x, P_y we write P^1, P^2, using superscripts, and call these quantities the contravariant components of the vector **P**. For the second set of components we write

$$P_x + P_y\cos\alpha = P_1,$$

$$P_y + P_x\cos\alpha = P_2$$

and call these the covariant components of **P**. The quantities P^n are vector components in the familiar sense of the word. When we multiply them with the unit vectors $\mathbf{i}_x = \mathbf{i}_1$ and $\mathbf{i}_y = \mathbf{i}_2$ and add the products, we obtain the vector **P**:

$$\mathbf{P} = P^1\mathbf{i}_1 + P^2\mathbf{i}_2 = \sum P^n\mathbf{i}_n. \tag{1.7a}$$

The covariant components can be added in a similar manner if we interpret them as shown in Figure 1.3. This figure contains, besides the axes 1 and 2, two other axes, which are at right angles to them. On these we project **P** by the usual parallelogram construction to obtain components of the magnitude

$$\frac{P_x + P_y\cos\alpha}{\sin\alpha} = \frac{P_1}{\sin\alpha}$$

and

$$\frac{P_y + P_x\cos\alpha}{\sin\alpha} = \frac{P_2}{\sin\alpha}.$$

When we interpret them as vectors, they add up to form **P**. We write them

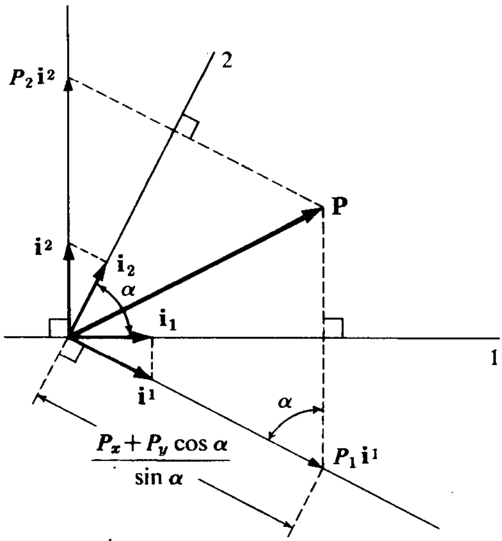

FIGURE 1.3 *Covariant and contravariant components of a vector.*

using reference vectors $\mathbf{i}^1$, $\mathbf{i}^2$, which are not unit vectors, but have the absolute value $1/\sin\alpha$. Then we can write

$$\mathbf{P} = P_1\mathbf{i}^1 + P_2\mathbf{i}^2 = \sum_m P_m\mathbf{i}^m \tag{1.7b}$$

as a second component representation of the vector **P**.

The idea explained here in two dimensions may easily be extended to three (and even more) dimensions. We choose an arbitrary set of three unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, called a reference frame. Then we resolve an arbitrary vector $\mathbf{v}$ in the usual way into components along the directions of these unit vectors and write them as $v^n\mathbf{i}_n$, $n = 1, 2, 3$. The vector $\mathbf{v}$ is then the sum of these contravariant components

$$\mathbf{v} = \sum_n v^n\mathbf{i}_n . \tag{1.8}$$

Next, we choose vectors $\mathbf{i}^m$, which satisfy the condition

$$\mathbf{i}^m \cdot \mathbf{i}_n = \delta^m_n, \tag{1.9}$$

where δ^m_n is another way of writing the Kronecker symbol δ_{mn} defined in (1.4). Each of the vectors $\mathbf{i}^m$ defined by (1.9) is at right angles to the vectors $\mathbf{i}_n$ with $n \neq m$ and of such magnitude that its absolute value $|\mathbf{i}^n|$ is the reciprocal of cos $(\mathbf{i}_n, \mathbf{i}^n)$. We may now resolve $\mathbf{v}$ in components in the directions of the vectors $\mathbf{i}^m$ and write

$$\mathbf{v} = \sum v_m \mathbf{i}^m. \tag{1.10}$$

Since, in general, $|\mathbf{i}^m| \neq 1$, the covariant components v_m are not simply the absolute values of the component vectors $v_m \mathbf{i}^m$.

When we now consider any two vectors $\mathbf{u}$ and $\mathbf{v}$, we may resolve one of them into contravariant components according to (1.8) and the other one into covariant ones according to (1.10):

$$\mathbf{u} = \sum_n u^n \mathbf{i}_n, \qquad \mathbf{v} = \sum_m v_m \mathbf{i}^m.$$

The dot product is then

$$\mathbf{u} \cdot \mathbf{v} = \sum_n \sum_m u^n v_m \mathbf{i}_n \cdot \mathbf{i}^m = \sum_n \sum_m u^n v_m \delta_n^m. \tag{1.11}$$

The double sum contains all possible combinations of n and m, nine terms all together. However, only in the three terms for which $n = m$, does the Kronecker symbol $\delta_n^m = 1$, while for the other six it equals zero. We may, therefore, write

$$\mathbf{u} \cdot \mathbf{v} = \sum_n u^n v_n = u^1 v_1 + u^2 v_2 + u^3 v_3, \tag{1.12}$$

which shows that also in skew rectilinear coordinates the formula for the dot product is as simple as (1.5), if for one vector we use the contravariant components and for the other the covariant ones.

We may now do the final step to build up the notation to be used with vectors and tensors. It will turn out that we always have to deal with sums over some index which appears twice in each term, once as a superscript in a contravariant component and once as a subscript indicating a covariant component. We shall in all these cases omit the summation sign and use the

> SUMMATION CONVENTION: Whenever the same Latin letter (say n) appears in a product once as a subscript and once as a superscript, it is understood that this means a sum of all terms of this kind (i.e. for $n = 1$, 2, 3).

With this convention we rewrite (1.12) as

$$\mathbf{u} \cdot \mathbf{v} = u^n v_n \tag{1.13}$$

and (1.11) as

$$\mathbf{u} \cdot \mathbf{v} = u^n v_m \mathbf{i}_n \cdot \mathbf{i}^m = u^n v_m \delta_n^m, \tag{1.14}$$

implying in this case a summation over all n and over all m.

Since in the result of such a summation the summation index no longer appears, it does not matter which letter we use for it. Such an index is called a dummy index, and when necessary, we may change the letter used for it from one equation to the next or from the left-hand side to the right-hand side of the same equation. It will often be necessary to do so because we must avoid making the summation convention unclear by using the same letter for two sums.

1.2. Base Vectors, Metric Tensor

In (1.7a) we used unit vectors $\mathbf{i}_n$ as a base for defining the contravariant components, but in (1.7b) we found it necessary to choose vectors $\mathbf{i}^n$, which do not have unit magnitude. We broaden our experience by considering a vector in a polar coordinate system, Figure 1.4. As a specimen vector we choose a line element $d\mathbf{s}$. Defining unit vectors $\mathbf{i}_1, \mathbf{i}_2$ in the direction of increasing coordinates, we can write

$$d\mathbf{s} = \mathbf{i}_1\, dr + \mathbf{i}_2 r\, d\theta. \tag{1.15}$$

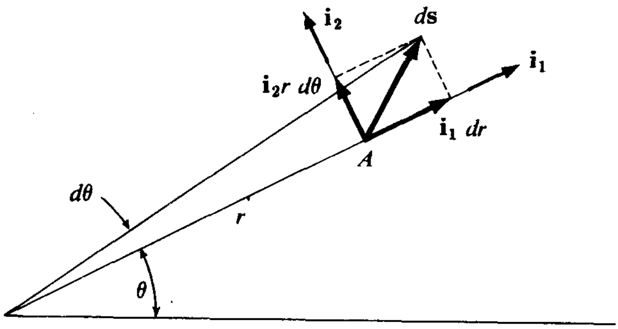

FIGURE 1.4 *Base vectors in polar coordinates.*

Here, as everywhere, we want to consider the differentials of the coordinates as the contravariant components of the line element vector $d\mathbf{s}$:

$$dr = dx^1, \qquad d\theta = dx^2.$$

It is then necessary that, instead of unit vectors, we use the coefficients of these differentials in (1.15) as base vectors:

$$\mathbf{g}_1 = \mathbf{i}_1, \qquad \mathbf{g}_2 = \mathbf{i}_2 r.$$

We call them the contravariant *base vectors* and rewrite (1.15) in the form

$$d\mathbf{s} = \mathbf{g}_1\, dx^1 + \mathbf{g}_2\, dx^2 = \mathbf{g}_i\, dx^i. \tag{1.16}$$

While $\mathbf{g}_1$ is still a unit vector, $\mathbf{g}_2$ has the absolute value r and is not even dimensionless, as the unit vectors are. We see also that, different from rectilinear coordinates, these base vectors are not constant, but depend on the coordinates of the point A, for which they have been defined. The directions of $\mathbf{g}_1$ and $\mathbf{g}_2$ depend on θ and the magnitude of $\mathbf{g}_2$ depends on r.

We generalize (1.16) by extending it to an arbitrary (possibly curvilinear) three-dimensional coordinate system x^i ($i = 1, 2, 3$). At any point A we choose three vectors $\mathbf{g}_i$ of such direction and magnitude that the line element vector

$$d\mathbf{s} = \mathbf{g}_i \, dx^i. \tag{1.17}$$

Now consider the position vector $\mathbf{r}$ leading from a fixed point O (possibly the origin of the coordinates) to the point A. The line element $d\mathbf{s}$ is the increment of $\mathbf{r}$ connected with the transition to an adjacent point, $d\mathbf{s} = d\mathbf{r}$. We can write this increment in the form

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial x^i} \, dx^i,$$

where again the summation convention is to be applied (accepting the superscript in the denominator in lieu of the required subscript). Comparing this expression with (1.17), we see that

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial x^i}. \tag{1.18}$$

We apply the base vectors $\mathbf{g}_i$ defined by (1.17) or (1.18) to all vectors associated with the point A. As an example, a force $\mathbf{P}$ acting at this point is written as

$$\mathbf{P} = \mathbf{g}_i P^i; \tag{1.19}$$

any of the components P^i has the dimension of a force if the corresponding $\mathbf{g}_i$ is dimensionless [as $\mathbf{g}_1$ in (1.16)] and otherwise has such a dimension that its product with $\mathbf{g}_i$ is a force.

A second set of base vectors $\mathbf{g}^j$ is defined by an equation similar to (1.9):

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j. \tag{1.20}$$

Each vector $\mathbf{g}^j$ is at right angles to all vectors $\mathbf{g}_i$ for which $i \neq j$ and has such magnitude (and such a dimension) that its dot product with $\mathbf{g}_j$ equals unity. This defines completely the vectors $\mathbf{g}^j$, which are called the contravariant base vectors. They may be used to define covariant components P_j of any vector $\mathbf{P}$:

$$\mathbf{P} = \mathbf{g}^j P_j. \tag{1.21}$$

When we apply the definitions (1.19) and (1.21) to any two vectors **u** and **v**, we may write their dot product as

$$\mathbf{u} \cdot \mathbf{v} = u^i \mathbf{g}_i \cdot v_j \mathbf{g}^j = u^i v_j \mathbf{g}_i \cdot \mathbf{g}^j = u^i v_j \delta_i^j = u^i v_i \tag{1.22a}$$

since

$$v_j \delta_i^j = v_i$$

and in the alternate form

$$\mathbf{u} \cdot \mathbf{v} = u_i \mathbf{g}^i \cdot v^j \mathbf{g}_j = u_i v^j \delta_j^i = u_i v^i. \tag{1.22b}$$

Every vector can be resolved into covariant or into contravariant components. When we try to write the covariant base vector $\mathbf{g}_1$ in contravariant components, we have

$$\mathbf{g}_1 = \mathbf{g}_1 \cdot 1 + \mathbf{g}_2 \cdot 0 + \mathbf{g}_3 \cdot 0,$$

i.e. a triviality. No matter what the actual magnitude of $\mathbf{g}_1$ is, it always has the components (1, 0, 0) in the system of base vectors $\mathbf{g}_i$. However, when we resolve a covariant base vector into covariant components, we are led to a set of new, important quantities:

$$\mathbf{g}_i = g_{ij} \mathbf{g}^j. \tag{1.23a}$$

The entity of the nine quantities g_{ij} thus defined is called the *metric tensor* and the individual g_{ij} are its covariant components. The meaning which stands behind this terminology will become clear when we study the tensor concept (see p. 15).

In analogy to (1.23a), we may resolve $\mathbf{g}^i$ into contravariant components,

$$\mathbf{g}^i = g^{ij} \mathbf{g}_j \tag{1.23b}$$

and thus define contravariant components of the metric tensor.

Let us now consider dot products of base vectors of the same set:

$$\mathbf{g}_i \cdot \mathbf{g}_j = g_{ik} \mathbf{g}^k \cdot \mathbf{g}_j = g_{ik} \delta_j^k = g_{ij} \tag{1.24a}$$

or

$$\mathbf{g}^i \cdot \mathbf{g}^j = g^{ik} \mathbf{g}_k \cdot \mathbf{g}^j = g^{ik} \delta_k^j = g^{ij}. \tag{1.24b}$$

Since the two factors in a dot product may be interchanged, it follows that

$$g_{ij} = g_{ji}, \qquad g^{ij} = g^{ji}. \tag{1.25}$$

Equations (1.24) may be used as the definitions of g_{ij} and g^{ij}. If this is done, (1.23) must be derived from them. This can be done in the following way: Tentatively, let

$$\mathbf{g}_i = a_{ij} \mathbf{g}^j$$