Tracts in Mathematics 24

Hans Triebel

Hybrid Function Spaces, Heat and Navier–Stokes Equations

European Mathematical Society

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Preface

This book is the continuation of [T13]. Our aim is twofold. First we develop the theory of *hybrid spaces* $L^r A^s_{p,q}(\mathbb{R}^n)$ which are between the nowadays well-known *global spaces* $A^s_{p,q}(\mathbb{R}^n)$ with $A \in \{B, F\}$ and their localization (or Morreyfication) $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ as considered in detail in [T13]. Spaces $A^s_{p,q}(\mathbb{R}^n)$ cover (fractional) Sobolev spaces, (classical) Besov spaces and Hölder-Zygmund spaces, whereas local Morrey spaces $\mathcal{L}^r_p(\mathbb{R}^n)$ are special cases of the *local spaces* $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$. In [T13] we applied the theory of spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ to nonlinear heat equations and Navier-Stokes equations. But this caused some problems which will be discussed in the Introduction (Chapter 1) below. It came out quite recently that it is more natural in this context to switch from *local spaces* $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ to *hybrid spaces* $L^r A^s_{p,q}(\mathbb{R}^n)$. This again will be illuminated in the Introduction below.

It is the second aim of this book to apply the theory of global spaces $A^s_{p,q}(\mathbb{R}^n)$ and hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$ to the Navier-Stokes equations

$$\partial_t u + (u, \nabla)u - \Delta u + \nabla P = 0 \qquad \text{in } \mathbb{R}^n \times (0,T), \tag{0.1}$$

$$\operatorname{div} u = 0 \qquad \text{in } \mathbb{R}^n \times (0,T), \tag{0.2}$$

$$u(\cdot, 0) = u_0 \qquad \text{in } \mathbb{R}^n, \tag{0.3}$$

in the version of

$$\partial_t u - \Delta u + \mathbb{P} \operatorname{div}(u \otimes u) = 0 \qquad \text{in } \mathbb{R}^n \times (0,T), \tag{0.4}$$

$$u(\cdot, 0) = u_0 \qquad \text{in } \mathbb{R}^n, \tag{0.5}$$

reduced to the scalar nonlinear heat equations

$$\partial_t v - D\, v^2 - \Delta v = 0 \qquad \text{in } \mathbb{R}^n \times (0,T), \tag{0.6}$$

$$v(\cdot, 0) = v_0 \qquad \text{in } \mathbb{R}^n, \tag{0.7}$$

where $0 < T \le \infty$. Here $u(x,t) = \big(u^1(x,t), \ldots, u^n(x,t)\big)$ in (0.1)–(0.5) is the unknown velocity and $P(x,t)$ the unknown (scalar) pressure, whereas $v(x,t)$ in (0.6), (0.7) is a scalar function, $2 \le n \in \mathbb{N}$. Recall $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ if $j = 1, \ldots, n$,

$$\big[(u, \nabla)u\big]^k = \sum_{j=1}^n u^j \partial_j u^k, \qquad k = 1, \ldots, n, \tag{0.8}$$

$$\operatorname{div} u = \sum_{j=1}^n \partial_j u^j, \qquad \nabla P = (\partial_1 P, \ldots, \partial_n P), \tag{0.9}$$

and by (0.2)

$$(u, \nabla)u = \operatorname{div}(u \otimes u), \qquad \operatorname{div}(u \otimes u)^k = \sum_{j=1}^n \partial_j (u^j u^k). \tag{0.10}$$

Furthermore, $\mathbb{P}$ is the Leray projector

$$(\mathbb{P}f)^k = f^k + R_k \sum_{j=1}^{n} R_j\, f^j, \qquad k = 1, \dots, n, \tag{0.11}$$

based on the (scalar) Riesz transforms

$$R_k g(x) = i\Big(\frac{\xi_k}{|\xi|}\widehat{g}\Big)^{\vee}(x) = c_n \lim_{\varepsilon\downarrow 0} \int_{|y|\ge\varepsilon} \frac{y_k}{|y|^{n+1}}\, g(x-y)\,\mathrm{d}y, \quad x \in \mathbb{R}^n. \tag{0.12}$$

In (0.4), (0.5) there is no need to care about (0.2) any longer. But if, in addition, $\operatorname{div} u_0 = 0$ then $\operatorname{div} u = 0$ in our context (mild solutions based on fixed point assertions). In the scalar equation (0.6) we used the abbreviation

$$D f = \sum_{j=1}^{n} \partial_j f. \tag{0.13}$$

As mentioned above we dealt in [T13] with the above equations in the context of the global spaces $A^s_{p,q}(\mathbb{R}^n)$. Rigorous reduction of (0.1)–(0.3) to (0.4), (0.5) and finally to (0.6), (0.7) requires a detailed study of the nonlinearity $u \mapsto u^2$ and of boundedness of Riesz transforms in the underlying spaces. In [T13] we tried to extend this theory to some local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$. But one needs some modifications, especially a replacement of the Riesz transforms by some truncated Riesz transforms. In the Introduction below we repeat the above considerations in greater details and discuss in particular this somewhat disturbing (but unavoidable) point. The hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$ preserve many desirable properties of the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ but avoid the above-indicated shortcomings. They are between global spaces and local spaces, which may justify calling them hybrid spaces. They coincide with the well-studied spaces $A^{s,\tau}_{p,q}(\mathbb{R}^n)$, $\tau = \frac{1}{p} + \frac{r}{n}$, including the global spaces $A^{s,0}_{p,q}(\mathbb{R}^n) = A^s_{p,q}(\mathbb{R}^n)$ as special cases.

Chapter 1 is the announced Introduction where we return to the above description in greater details and with some references. Chapter 2 deals with local and global Morrey spaces $\mathcal{L}^r_p(\mathbb{R}^n)$, $L^r_p(\mathbb{R}^n)$, their duals and preduals and, in particular, with the question whether the Riesz transforms R_k in (0.12) are bounded maps in these spaces and what they look like. This chapter is self-contained and we hope that it is of interest for researchers in this field. In Chapter 3 we develop the theory of the hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$ as needed for our above-outlined purposes. It comes out that many basic properties for the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ can be transferred easily from [T13] to the hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$. We concentrate on some new aspects which will be crucial in the context described above. Similarly we carry over and complement in Chapter 4 the theory of heat equations in the global spaces $A^s_{p,q}(\mathbb{R}^n)$ as developed in [T13] to the hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$. Chapter 5 deals with Navier-Stokes equations especially in the version (0.4), (0.5) in hybrid spaces. Then one is in a rather comfortable position, clipping together related assertions of the two preceding chapters. But again we add a few new aspects. In particular, if the admitted initial data are infrared-damped then the related local solutions of the Navier-Stokes equations can be extended globally in time. These considerations will be continued in Chapter 6 now specified to the global spaces $A^s_{p,q}(\mathbb{R}^n)$ and extended to the spaces $S^r_{p,q}A(\mathbb{R}^n)$ with dominating mixed smoothness. We discuss conditions for the initial data in terms of Haar wavelets, Faber bases, and sampling in connection with the hyperbolic cross, ensuring solutions of the Navier-Stokes equations which are global in time. Furthermore we add some comments about the influence of large Reynolds numbers. This chapter is largely independent of the preceding considerations.

We assume that the reader has a working knowledge about basic assertions for the spaces $A^s_{p,q}(\mathbb{R}^n)$. But to make this book independently readable we provide related notation, facts, and detailed references. Formulae are numbered within chapters. Furthermore in each chapter all definitions, theorems, propositions, corollaries and remarks are jointly and consecutively numbered. References are ordered by names, not by labels, which roughly coincide, but may occasionally cause minor deviations. The bracketed numbers following the items in the Bibliography mark the page(s) where the corresponding entry is quoted. All unimportant positive constants will be denoted by c (with additional marks if there are several c's in the same formula). To avoid any misunderstanding we fix our use of $\sim$ (equivalence) as follows. Let I be an arbitrary index set. Then

$$a_i \sim b_i \quad \text{for} \quad i \in I \qquad \text{(equivalence)} \tag{0.14}$$

for two sets of positive numbers $\{a_i : i \in I\}$ and $\{b_i : i \in I\}$ means that there are two positive numbers c_1 and c_2 such that

$$c_1\, a_i \le b_i \le c_2\, a_i \qquad \text{for all } i \in I.$$

Contents

Chapter 1
Introduction

In [T13] we dealt with the Navier-Stokes equations

$$\partial_t u + (u,\nabla)u - \Delta u + \nabla P = 0 \qquad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1.1}$$
$$\operatorname{div} u = 0 \qquad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1.2}$$
$$u(\cdot,0) = u_0 \qquad \text{in } \mathbb{R}^n, \tag{1.3}$$

where $u(x,t) = \big(u^1(x,t),\ldots,u^n(x,t)\big)$ is the unknown velocity and $P(x,t)$ the unknown (scalar) pressure, $2 \le n \in \mathbb{N}$. Recall $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ if $j = 1,\ldots,n$, and that the vector-function $(u,\nabla)u$ has the components

$$\big[(u,\nabla)u\big]^k = \sum_{j=1}^n u^j \partial_j u^k, \qquad k = 1,\ldots,n, \tag{1.4}$$

whereas, as usual,

$$\operatorname{div} u = \sum_{j=1}^n \partial_j u^j, \qquad \nabla P = (\partial_1 P,\ldots,\partial_n P). \tag{1.5}$$

By (1.2) one has

$$(u,\nabla)u = \operatorname{div}(u\otimes u), \qquad \operatorname{div}(u\otimes u)^k = \sum_{j=1}^n \partial_j (u^j u^k). \tag{1.6}$$

This reduces (1.1)–(1.3), now in the strip $\mathbb{R}^n \times (0,T)$ with $T > 0$, to

$$\partial_t u - \Delta u + \mathbb{P}\operatorname{div}(u\otimes u) = 0 \qquad \text{in } \mathbb{R}^n \times (0,T), \tag{1.7}$$
$$u(\cdot,0) = u_0 \qquad \text{in } \mathbb{R}^n. \tag{1.8}$$

Here $\mathbb{P}$ is the Leray projector,

$$(\mathbb{P}f)^k = f^k + R_k \sum_{j=1}^n R_j f^j, \qquad k = 1,\ldots,n, \tag{1.9}$$

based on the (scalar) Riesz transforms R_k,

$$R_k g(x) = i\Big(\frac{\xi_k}{|\xi|}\widehat{g}\Big)^\vee(x) = c_n \lim_{\varepsilon\downarrow 0}\int_{|y|\ge\varepsilon} \frac{y_k}{|y|^{n+1}}\, g(x-y)\,\mathrm{d}y, \quad x\in\mathbb{R}^n. \tag{1.10}$$

In (1.7), (1.8) there is no need to care about (1.2) any longer. But if in addition $\operatorname{div} u_0 = 0$ then $\operatorname{div} u = 0$ in our context (mild solutions based on fixed point arguments). This well-known reduction of (1.1)–(1.3) to (1.7), (1.8) may also be found in [T13, Section 6.1.3, pp. 196-198]. The vector equation (1.7), (1.8) can be reduced to the nonlinear scalar heat equation

$$\partial_t u(x,t) - D\, u^2(x,t) - \Delta u(x,t) = 0, \qquad x \in \mathbb{R}^n,\ 0 < t < T, \tag{1.11}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n, \tag{1.12}$$

on the one hand and the mapping properties of R_j and $\mathbb{P}$ in the considered function spaces on the other hand. Here

$$D f = \sum_{j=1}^{n} \partial_j f. \tag{1.13}$$

We dealt with the Cauchy problem (1.11), (1.12) in the context of *local spaces* $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$, [T13, Theorem 5.24, p. 183], and of *global spaces* $A^s_{p,q}(\mathbb{R}^n)$, [T13, Theorem 5.36, p. 189], under the crucial assumption that the underlying spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ and $A^s_{p,q}(\mathbb{R}^n)$ are multiplication algebras. This is ensured if $s + r > 0$ for local spaces and $s > n/p$ (and some limiting spaces with $s = n/p$) for global spaces. The reduction of (1.7), (1.8) to (1.11), (1.12) requires in addition that the Riesz transforms R_j are linear and bounded maps in the underlying spaces. This applies to the global spaces

$$A^s_{p,q}(\mathbb{R}^n), \qquad 1 < p < \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}, \tag{1.14}$$

[T13, Theorem 1.25, p. 17] where the additional restriction $1 < q < \infty$ for F-spaces mentioned there is not necessary (as a consequence of Theorem 3.52 below). Then one obtains satisfactory solutions for (1.7), (1.8) in the global spaces

$$A^s_{p,q}(\mathbb{R}^n), \qquad 1 < p < \infty, \quad 1 \le q \le \infty, \quad s > n/p, \tag{1.15}$$

(and some limiting cases with $s = n/p$). We refer the reader to [T13, Theorem 6.7, p. 203] (where $1 < q < \infty$ for F-spaces can be replaced by $1 \le q \le \infty$ as covered by Corollary 5.4 below). We could not find a counterpart in terms of the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ and replaced as a substitute the Leray projector $\mathbb{P}$ in (1.7) by the truncated Leray projector $\mathbb{P}_{\psi^2}$ based on the truncated Riesz transforms

$$R_{\psi,k} f = i\Big(\psi\, \frac{\xi_k}{|\xi|}\, \widehat{f}\Big)^\vee, \qquad k = 1, \ldots, n, \tag{1.16}$$

where

$$\psi \in C^\infty(\mathbb{R}^n), \quad \psi(x) = 0 \text{ if } |x| \le 1/2 \quad \text{and} \quad \psi(y) = 1 \text{ if } |y| \ge 1, \tag{1.17}$$

[T13, pp. 199/200, Theorem 6.10, p. 205]. Hence, one removes the infrared (or low frequency) part of solutions of (1.7), (1.8). This point has also been discussed in

[T13, p. 193, 199-201]. At that time we tried to find a way to deal with Navier-Stokes equations or with (1.7), (1.8) also in the context of the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$. But it came out quite recently that the Riesz transform (1.10) cannot be extended from $D(\mathbb{R}^n)$ or $S(\mathbb{R}^n)$ to a linear and bounded operator acting in the local Morrey spaces $\mathcal{L}^r_p(\mathbb{R}^n) = \mathcal{L}^r L_p(\mathbb{R}^n)$, $1 < p < \infty$, $-n/p \le r < 0$, [RoT13, Theorem 1.1(i)]. We refer the reader also to Theorem 2.22 and Remark 2.23 below. On the one hand one can take this observation as a justification of the above truncation. But on the other hand one knows now that R_k are linear and bounded maps,

$$R_k : \quad \mathring{L}^r_p(\mathbb{R}^n) \hookrightarrow \mathring{L}^r_p(\mathbb{R}^n) \quad \text{and} \quad L^r_p(\mathbb{R}^n) \hookrightarrow L^r_p(\mathbb{R}^n), \tag{1.18}$$

$1 < p < \infty$, $-n/p \le r < 0$, in the global Morrey spaces $L^r_p(\mathbb{R}^n) = L^r L_p(\mathbb{R}^n)$ and in the completion of $S(\mathbb{R}^n)$ in $L^r_p(\mathbb{R}^n)$, denoted as $\mathring{L}^r_p(\mathbb{R}^n)$, [RoT13, Theorem 1.1], Theorem 2.22 and Remark 2.23 below. We refer the reader also to [RoT14]. It is crucial for us and the main motivation of this book that (1.18) can be extended to some *hybrid spaces* $L^r A^s_{p,q}(\mathbb{R}^n)$ (being smaller than the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$). As far as properties are concerned these spaces are between *local* and *global* spaces. This may justify calling them *hybrid* spaces. In particular if

$$1 < p < \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R} \quad \text{and} \quad -n/p \le r < 0, \tag{1.19}$$

then one has by Theorem 3.52 below

$$R_k : \quad L^r A^s_{p,q}(\mathbb{R}^n) \hookrightarrow L^r A^s_{p,q}(\mathbb{R}^n), \qquad k = 1, \ldots, n, \tag{1.20}$$

whereas the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ do not have this property. In addition $L^r A^s_{p,q}(\mathbb{R}^n)$ are multiplication algebras if $s + r > 0$ (as their local counterparts $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$). Then one can extend a corresponding theory for the nonlinear heat equations (1.11), (1.12), now in terms of the hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$, to the Navier-Stokes equations. We tried to find in [T13] related assertions in the context of the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$. Now it is clear that this is impossible, but it is also clear that one has a satisfactory theory with hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$ in place of the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$. This extends corresponding assertions from $A^s_{p,q}(\mathbb{R}^n) = L^{-n/p} A^s_{p,q}(\mathbb{R}^n)$ to $L^r A^s_{p,q}(\mathbb{R}^n)$.

Chapter 2 deals mainly with local and global Morrey spaces $\mathcal{L}^r_p(\mathbb{R}^n)$, $\mathring{\mathcal{L}}^r_p(\mathbb{R}^n)$, $L^r_p(\mathbb{R}^n)$, $\mathring{L}^r_p(\mathbb{R}^n)$ and their (pre)duals. We follow closely [RoT13, RoT14] complemented by

$$L^r_p(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n, \mu_\alpha), \quad \mu_\alpha = w_\alpha \mu_L, \quad 1 < p < \infty, \quad -n/p < r < 0, \tag{1.21}$$

where μ_L is the Lebesgue measure and $w_\alpha(x) = (1 + |x|^2)^{\alpha/2}$ with $-n < \alpha < -n - rp$ is a Muckenhoupt weight $w_\alpha \in \mathcal{A}_p(\mathbb{R}^n)$. Then $R_k g(x)$ according to (1.10) is well-defined for $x \in \mathbb{R}^n$ a.e., also in its integral version. Finally we characterize some of these spaces in terms of Haar wavelets. In Chapter 3 we introduce the hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$ and collect some basic properties needed later on. This can be done largely in the same way as in [T13] for the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ mostly without additional efforts. Only occasionally we add a further argument. We observe that

$$L^r A^s_{p,q}(\mathbb{R}^n) = A^{s,\tau}_{p,q}(\mathbb{R}^n) \qquad \text{with} \quad \tau = \frac{1}{p} + \frac{r}{n} \tag{1.22}$$

for all admitted parameters s, p, q and $-n/p \le r < \infty$. The spaces $A^{s,\tau}_{p,q}(\mathbb{R}^n)$ have been studied in great detail in the book [YSY10], the survey [Sic12] and the underlying papers. There one finds many other properties which will not be repeated here. One may also consult [T13, pp. 38/39, Section 2.7.3, pp. 101-107]. There is one crucial exception needed to prove (1.20). Then we rely on

$$\|f \,|L^r A^s_{p,q}(\mathbb{R}^n)\| \sim \|f \,|L^r \dot{A}^s_{p,q}(\mathbb{R}^n)\| + \|f \,|L^r_p(\mathbb{R}^n)\| \tag{1.23}$$

if

$$1 < p < \infty, \quad 0 < q \le \infty, \quad s > 0, \quad -n/p \le r < 0. \tag{1.24}$$

Here $L^r \dot{A}^s_{p,q}(\mathbb{R}^n)$ are homogeneous hybrid spaces (we do not need the homogeneous spaces themselves but only their homogeneous norms in the context of the inhomogeneous spaces $L^r A^s_{p,q}(\mathbb{R}^n)$). For these homogeneous spaces (or their norms) one has the Fourier multiplier assertion

$$\|(h\widehat{f})^\vee |L^r \dot{A}^s_{p,q}(\mathbb{R}^n)\| \le c \Big(\sup_{|\alpha|\le k, x\in\mathbb{R}^n} |x|^{|\alpha|} \big| D^\alpha h(x)\big|\Big) \|f \,|L^r \dot{A}^s_{p,q}(\mathbb{R}^n)\| \tag{1.25}$$

of Michlin type with $k \in \mathbb{N}$ sufficiently large (specified later on). This is essentially covered by [YaY10, Theorem 4.1, p. 3819]. We refer also to [YYZ12, Theorem 1.5, p. 6] and the recent survey [YaY13a]. This can be applied to R_k with $h = \xi_k/|\xi|$. Then (1.20) with (1.19) follows essentially from (1.23) and (1.18), (1.25). This may be considered as the basic observation of what follows. Afterwards we return in Chapter 4 to the nonlinear heat equations (1.11), (1.12) and transfer assertions available so far in the context of the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ to their hybrid counterparts $L^r A^s_{p,q}(\mathbb{R}^n)$ (again essentially without any additional efforts) complemented by some new observations. In Chapter 5 we deal with the Navier-Stokes equations (1.7), (1.8) in hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$ extending a corresponding theory in [T13] for the spaces $A^s_{p,q}(\mathbb{R}^n) = L^{-n/p} A^s_{p,q}(\mathbb{R}^n)$ to $L^r A^s_{p,q}(\mathbb{R}^n)$. This extension applies not only to the obtained assertions, but also to the underlying technicalities. In particular (1.23) is the Morreyfied version of

$$\|f \,|A^s_{p,q}(\mathbb{R}^n)\| \sim \|f \,|\dot{A}^s_{p,q}(\mathbb{R}^n)\| + \|f \,|L_p(\mathbb{R}^n)\| \tag{1.26}$$

if

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s > \sigma_p = n\big(\tfrac{1}{p} - 1\big)_+, \tag{1.27}$$

[T92, Theorem 2.3.3, p. 98]. Furthermore, (1.25) with

$$\dot{A}^s_{p,q}(\mathbb{R}^n) = L^{-n/p} \dot{A}^s_{p,q}(\mathbb{R}^n) \tag{1.28}$$

is covered by [T83, Theorem 5.2.2, p. 241]. We refer the reader also to [T13, Theorem 1.25, p. 17]. The final Chapter 6 is to some extent independent of the main bulk of this book. It deals with Haar wavelets, Faber bases and sampling in the context of the hyperbolic cross and spaces with dominating mixed smoothness and their relations to solutions of Navier-Stokes equations, global in time, for large initial data.

Chapter 2
Morrey spaces

2.1 Introduction

This chapter deals with local Morrey spaces $\mathcal{L}^r_p(\mathbb{R}^n)$ and global Morrey spaces $L^r_p(\mathbb{R}^n)$ as well as their preduals $\mathcal{H}^\varrho L_p(\mathbb{R}^n)$, $H^\varrho L_p(\mathbb{R}^n)$ in the framework of tempered distributions $S'(\mathbb{R}^n)$. This requires some restrictions for the parameters, typically $1 < p < \infty$. We are especially interested in duality properties and embeddings between these spaces and in relations to distinguished Besov spaces. It is our intention to present the material as self-contained as possible and to illuminate the somewhat tricky (topological) background to a larger extent than usually done in the literature. This requires that we include some basic material and a few so-called well-known properties for which we could not find proofs in the literature. A typical example is the claim that Morrey spaces are non-separable. We give a short proof. As a by-product one obtains in one line the highly desirable (and well-known) assertion that neither $D(\mathbb{R}^n)$, nor $S(\mathbb{R}^n)$, nor distinguished Lebesgue spaces are dense in Morrey spaces.

The second main aim of this chapter is the study of mapping properties of Calderón-Zygmund operators in Morrey spaces and their preduals. The Riesz transforms (1.10) are distinguished cases and the mapping property (1.18) will be of great service for us in later chapters. It will be crucial to justify (1.20) based on (1.23), (1.25).

There are apparently no books or up-to-date comprehensive surveys dealing with Morrey spaces and their (pre)duals especially in the limelight of Harmonic Analysis. Basic material about Morrey-Campanato spaces may be found in [KJF77, Chapter 4], taken over to the new edition [PKJF13, Chapter 5]. We do not deal with the numerous modifications and generalizations of (Campanato)-Morrey spaces. The interested reader may consult the overview [RSS13] where one finds also references to related classical and recent papers. In the last few years remarkable progress has been made to raise Morrey spaces in Harmonic Analysis to the same level as Lebesgue spaces. The most advanced paper in this direction is [AdX12], based on [AdX04], and the literature mentioned there spanning a period of several decades. This chapter may also be considered as a contribution to these recent developments providing some background material based on new proofs. In [T13, Chapter 3] we dealt with Morrey-Campanato spaces $\mathcal{L}^r_p(\mathbb{R}^n)$ in the larger context of so-called local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$. This will now be complemented, identifying global Morrey spaces $L^r_p(\mathbb{R}^n)$ with special hybrid spaces. As a consequence we characterize Morrey spaces in terms of Haar wavelets.

In Section 2.2 we collect some definitions and comment on notation and the range of the admitted parameters. Section 2.3 deals with embeddings of Morrey spaces in the framework of $\big(S(\mathbb{R}^n), S'(\mathbb{R}^n)\big)$, complemented by the non-separability of $\mathcal{L}^r_p(\mathbb{R}^n)$