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THE LOCATION OF CRITICAL POINTS
OF ANALYTIC AND HARMONIC FUNCTIONS

BY
J. L. WALSH
HARVARD UNIVERSITY

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PREFACE

Commencing with Gauss, numerous mathematicians have contributed to the study in the plane of the complex variable of the geometric relations between the zeros of a polynomial and those of its derivative. More generally there have been studied the relations between the zeros and poles of an arbitrary rational function and the zeros of its derivative, and an elaborate body of material has evolved which applies also to the critical points of an arbitrary analytic function, of Green's function, and of other harmonic functions. It is the purpose of the present volume to assemble and unify a large portion of this material, to make it available for study by the beginner or by the specialist, and for reference.

Results concerning even polynomials and rational functions are far too numerous to be completely treated here. The omission is not serious, for a general survey of the entire field of the geometry of zeros of polynomials has recently been written by Marden [1949]*, to which the reader may refer for broader perspective in that field. The present material has been chosen to emphasize (i) the determination of regions which are free from critical points (or alternately, which contain all critical points), rather than to study mere enumeration of critical points in a given region, and (ii) results for polynomials and rational functions which can be extended to and furnish a pattern for the cases of more general analytic functions and of harmonic functions. Our main problem, then, is the approximate determination of critical points—approximate not in the sense of computation which may be indefinitely refined, but in the sense of geometric limitation of critical points to easily constructed regions, preferably bounded by lines and circles. The point sets shown to contain the critical points are naturally defined in terms of the zeros of a given polynomial, in terms of the zeros and poles of a given rational or more general analytic function, and in terms of suitable level curves of a given harmonic function. The theorem of Lucas is typical in this field both as to general content and method of proof, and occupies a central position in the entire theory. Although there are close connections with topology, our methods are mainly study of a field of force, use of algebraic inequalities, analytic geometry, geometry of circles and plane curves, circle transformations, potential theory, and conformal mapping. So far as concerns rational functions, the methods are largely elementary, as seems to be in keeping with the nature of the problems.

We use the term *critical point* to include both zeros of the derivative of an analytic function and points where the two first partial derivatives of a harmonic function vanish. It is hardly necessary to emphasize the importance of critical points as such: 1) they are notable points in the behavior of the function, and in particular for a harmonic function are the multiple points of level curves and

* Dates in square brackets refer to the Bibliography.

their orthogonal trajectories (curves of steepest descent); for an analytic function $f(z)$ they are the multiple points of the loci $|f(z)| = \text{const}$ and $\arg [f(z)] = \text{const}$; 2) for an analytic function $f(z)$ they are the points where the conformality of the transformation $w = f(z)$ fails; 3) they are conformal invariants, of importance in numerous extremal problems of analytic functions and in the study of approximation by rational and other functions; 4) they are precisely the positions of equilibrium in a potential field of force due to a given distribution of matter or electricity; 5) they are stagnation points in a field of velocity potential. We obtain incidentally in the course of our work numerous results concerning the direction of the force or velocity corresponding to a given potential, but such results are secondary and are frequently implicit, being left to the reader for formulation.

The term *location* of critical points suggests the term *locus*, and ordinarily we determine an actual locus of critical points under suitable restrictions on the given function. In the former part of the book we take pains to indicate this property, but in the latter part leave to the reader the detailed discussion. Of course our entire problem, of determining easily found regions free from (or containing) critical points, is clearly a relative one, and it is a matter of judgment how far the theory should be developed. Convenience, simplicity, and elegance are our criteria, but the theory admits of considerable further development, especially in the use of algebraic curves of higher degree.

A large part of the material here set forth has not been previously published, except perhaps in summary form. Thus Chapter V considers in some detail the critical points of rational functions which possess various kinds of symmetry. Chapters VII–IX present a unified investigation of the critical points of harmonic functions, based on the study of a field of force due to a spread of matter on the boundary of a given region, a boundary which may consist of a finite number of arbitrary Jordan curves. The introduction and use of special loci called *W-curves* adds unity and elegance to much of our entire discussion.

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J. L. WALSH

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CHAPTER I

FUNDAMENTAL RESULTS

§1.1. Terminology. Preliminaries. We shall be concerned primarily with the plane of the complex variable $z = x + iy$; the plane may be either the *finite plane* (i.e. plane of finite points) or the *extended plane* (the finite plane with the adjunction of a single point at infinity). The extended plane is often studied by stereographic projection onto the sphere (*Neumann sphere*), on which the point at infinity is no longer exceptional. When we study primarily polynomials, we ordinarily deal with the finite plane except when we use an auxiliary inversion in a circle; when we deal with more general rational functions, or with harmonic functions, and with the non-integral linear transformation of the complex variable, we usually operate in the extended plane. We frequently identify a value of z and the point representing it; thus the term *non-real point* refers to a point representing a non-real value of z.

In the finite plane the term *circle* means simply circumference; in the extended plane the use of the term is ordinarily broadened to include straight line.

§1.1.1. Point set terminology. A *neighborhood* of a finite point is the interior of a circle with that point as center. A neighborhood of the point at infinity is the exterior of a circle of the finite plane, point at infinity included.

If S is a given point set, the point P is an *interior point* of S if some neighborhood of P contains only points of S, is a *boundary point* of S if every neighborhood of P contains both points of S and points not in S, and is an *exterior point* if some neighborhood of P contains no point of S. The point P is a *limit point* of S if every neighborhood of P contains an infinity of points of S. A set S is *open* if every point of S is an interior point of S; a set S is *closed* if it contains its limit points.

The *closure* of a set S consists of S plus its limit points; the closures of sets R, S, T are respectively denoted by $\bar{R}$, $\bar{S}$, $\bar{T}$.

A *Jordan arc* is the image of the closed line segment $0 \leq x \leq 1$ under a one-to-one continuous transformation, where continuity may be interpreted either in the finite plane or on the sphere, the latter being equivalent to the extended plane. A *Jordan curve* is similarly the image of a circumference under a one-to-one continuous transformation.

A *region* is an open set of which any two points can be joined by a Jordan arc consisting wholly of points of the set. A *closed region* is the closure of a region, and need not properly be a region, but the term *region* may be somewhat loosely applied to include closed region, and even arcs and points as degenerate closed regions. A *Jordan region* is a region bounded by a Jordan curve.

A *component* of a closed set S is a closed subset which cannot itself be separated, but which can be separated from all other points of S, by a Jordan curve disjoint from S.

A *Jordan configuration* is a point set consisting of a finite number of Jordan arcs.

§1.1.2. Function-theoretic preliminaries. A *polynomial in z of degree n* is a function which can be written as $a_0 z^n + a_1 z^{n-1} + \cdots + a_n$, $a_0 \neq 0$. A function $f(z)$ is *analytic* at a finite point z_0 if it possesses a continuous derivative at every point of a neighborhood of z_0, and is analytic at infinity if $f(1/z)$ is analytic at the origin.* A function is analytic in a region if it is analytic at every point of that region, and is assumed single-valued unless otherwise specified. A function $f(z)$ is *meromorphic* in a region if it is analytic there except perhaps for poles. A finite *critical point* of an analytic function $f(z)$ is a point z_0 at which the derivative $f'(z)$ vanishes (this notation for derivative will be used consistently); the *multiplicity* of z_0 as a critical point of $f(z)$ is the multiplicity of z_0 as a zero of $f'(z)$. The point at infinity is a critical point of an analytic function $f(z)$, and of order k, if the origin is a critical point of the function $f(1/z)$, and of order k. If an analytic function $f(z)$ has a pole at infinity, that point is not a critical point; if $f(z)$ is analytic there, the point at infinity is a critical point of order k if and only if $f'(z)$ has a zero at infinity of order $k + 2$.

A few well known theorems are of central importance in the sequel.

PRINCIPLE OF ARGUMENT. *Let R be a region whose boundary B consists of a finite number of mutually disjoint Jordan curves, and let the function f(z) be meromorphic in R, continuous on R + B in the closed neighborhood of B, different from zero on B. As z traces B in the positive sense (i.e. so that the region R is situated to the left of the forward moving observer), the net increase in* arg $[f(z)]$ *is* 2π *times the number of zeros minus the number of poles of f(z) interior to R.*

An application of this principle is

ROUCHÉ'S THEOREM. *Let the functions* $\varphi(z)$ *and* $\psi(z)$ *be analytic interior to a region R whose boundary B consists of a finite number of mutually disjoint Jordan curves, and let the functions* $\varphi(z)$ *and* $\psi(z)$ *be continuous in the closed region* $\bar{R}$, *with the relation* $|\varphi(z)| > |\psi(z)|$ *on B. Then the functions* $\varphi(z)$ *and* $\varphi(z) + \psi(z)$ *have the same number of zeros in R.*

Let z_0 be a point of R at which at least one of the functions $\varphi(z)$ and $\varphi(z) + \psi(z)$ vanishes; if z_0 is a zero of $\varphi(z) + \psi(z)$ of order m but not a zero of $\varphi(z)$, then z_0 is a zero of $f(z) \equiv [\varphi(z) + \psi(z)]/\varphi(z)$ of order m; if z_0 is a zero of $\varphi(z)$ of order n but not a zero of $\varphi(z) + \psi(z)$, then z_0 is a pole of $f(z)$ of order n; if z_0 is a zero of $\varphi(z) + \psi(z)$ of order m and a zero of $\varphi(z)$ of order n, then z_0 is respectively a zero of $f(z)$ of order $m - n$, or a non-zero point of analyticity of $f(z)$, or a pole of $f(z)$ of order $n - m$ according as we have $m > n$, or $m = n$, or $m < n$.

* We assume $f(1/z)$ to be defined for $z = 0$ so as to be continuous there if possible; a similar remark applies below at other isolated singularities.

Thus the number of zeros of $\varphi(z) + \psi(z)$ in R minus the number of zeros of $\varphi(z)$ in R equals the number of zeros of $f(z)$ minus the number of poles in R. On the other hand, as z traces each component of B the point $w = f(z)$ remains interior to the circle $|w - 1| = 1$, so the net total change in arg w is zero; the number of zeros of $f(z)$ minus the number of poles of $f(z)$ in R is zero.

HURWITZ'S THEOREM. *Let R be a region in which the functions $f_n(z)$ and $f(z)$ are analytic, continuous in $\bar{R}$, with $f(z)$ different from zero on the boundary B of R, and let the sequence $f_n(z)$ converge uniformly to $f(z)$ on $R + B$. Then for n greater than a suitably chosen N, the number of zeros of $f_n(z)$ in R is the same as the number of zeros of $f(z)$ in R.*

In the proof we assume, as we may do, that B consists of a finite number of mutually disjoint Jordan curves. If δ (> 0) is chosen so that $|f(z)| > \delta$ on B, we need merely choose N_δ so that $n > N_\delta$ implies $|f_n(z) - f(z)| < \delta$ on B, and apply Rouché's Theorem.

Hurwitz's Theorem applies not merely to the given region R, but also to a neighborhood $N(z_0)$ whose closure lies in R of an arbitrary zero z_0 of $f(z)$ in R, provided $f(z)$ does not vanish on the boundary of $N(z_0)$. If z_0 is a zero of $f(z)$ of order m, and if $N(z_0)$ contains no zero of $f(z)$ other than z_0, then n greater than a suitably chosen N_δ implies that $N(z_0)$ contains precisely m zeros of $f_n(z)$. It follows that if no function $f_n(z)$ vanishes identically, the limit points in R of the zeros of the functions $f_n(z)$ in R are precisely the zeros of $f(z)$ in R.

In particular, under the hypothesis of Hurwitz's Theorem with $f(z)$ not identically constant, the sequence $f'_n(z)$ converges uniformly to $f'(z)$ on any closed subset of R, so if z_0 is a finite or infinite critical point of $f(z)$ in R of order m, and if $N(z_0)$ is a neighborhood of z_0 to which B and all other critical points of $f(z)$ are exterior, then n greater than a suitably chosen N_δ implies that $N(z_0)$ contains precisely m critical points of $f_n(z)$; if no $f_n(z)$ is identically constant, the limit points in R of the critical points of the $f_n(z)$ are precisely the critical points of $f(z)$ in R; if $f'(z)$ and $f'_n(z)$ are continuous in $R + B$ and different from zero on B, and if $f'_n(z)$ converges uniformly in $R + B$, then for suitably large index the function $f_n(z)$ has precisely the same total number of critical points in R as does $f(z)$.

The zeros of an algebraic equation are continuous functions of the coefficients: if the variable polynomial $P_k(z)$ of bounded degree approaches the fixed polynomial $P(z)$ (not identically constant) in the sense that coefficients of $P_k(z)$ approach corresponding coefficients of $P(z)$, then each zero of $P(z)$ is approached by a number of zeros of $P_k(z)$ equal to its multiplicity; all other zeros of $P_k(z)$ become infinite. More explicitly, let

$$P(z) \equiv a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

be a polynomial not identically constant, and let

$$P_k(z) \equiv a_{0k} z^n + a_{1k} z^{n-1} + \cdots + a_{nk}$$

be a variable polynomial such that $a_{mk} \to a_m$ $(m = 0, 1, 2, \cdots, n)$ *as* k *becomes infinite. Let* $\alpha_1, \alpha_2, \cdots, \alpha_p$ *be the distinct zeros of* $P(z)$, *and let* $\epsilon > 0$ *be arbitrary,* $|\alpha_i - \alpha_j| > 2\epsilon$, $|\alpha_j| + \epsilon < 1/\epsilon$. *Then there exists* M_ϵ *such that* $k > M_\epsilon$ *implies that each neighborhood* $|z - \alpha_j| < \epsilon$ *contains precisely a number of zeros of* $P_k(z)$ *equal to the multiplicity of* α_j *as a zero of* $P(z)$, *and all other zeros of* $P_k(z)$ *lie exterior to the circle* $|z| = 1/\epsilon$.

The sequence $P_k(z)$ converges uniformly to the function $P(z)$ in any closed bounded region:

$$|P(z) - P_k(z)| \leqq |a_0 - a_{0k}| \cdot |z|^n + |a_1 - a_{1k}| \cdot |z|^{n-1} + \cdots + |a_n - a_{nk}|,$$

hence in the closed region $|z| \leqq 1/\epsilon$; the conclusion follows from Hurwitz's Theorem applied simultaneously to $|z| \leqq 1/\epsilon$ and to all the regions $|z - \alpha_j| \leqq \epsilon$. This method of proof is used by Bieberbach-Bauer [1928].

If we modify the hypothesis here so as to admit a polynomial $P(z)$ which is identically constant but not identically zero, the proof requires no modification, and shows that $k > M_\epsilon$ implies that all zeros of $P_k(z)$ lie exterior to $|z| = 1/\epsilon$.

A real function $u(x, y)$ or $u(z)$ is *harmonic* in a finite region if there it is continuous together with its first and second partial derivatives, and satisfies Laplace's equation. The function $u(z)$ is harmonic at a finite point if it is harmonic throughout a neighborhood of that point, and is harmonic at infinity if $u(1/z)$ is harmonic at the origin. If $u(x, y)$ is harmonic in a region R, it possesses a conjugate $v(x, y)$ there, which is single-valued if R is simply connected; the function $f(z) \equiv u(x, y) + iv(x, y)$ is then analytic in R. A finite *critical point* of $u(x, y)$ is a point at which $\partial u/\partial x$ and $\partial u/\partial y$ vanish, that is, a critical point of $f(z)$; it is sufficient if the directional derivatives of $u(x, y)$ in two essentially distinct (i.e. not the same nor opposite) directions vanish; the point at infinity is a critical point of $u(x, y)$ if and only if it is a critical point of $f(z)$. The *order* of a critical point of $u(x, y)$ is its order as a critical point of $f(z)$. If $z_0 = x_0 + iy_0$ is a finite critical point of $u(x, y)$ of order m, then all partial derivatives of $u(x, y)$ of orders $1, 2, \cdots, m$ vanish there, but not all partial derivatives of order $m + 1$.

If a function $f(z)$ is analytic at infinity, its derivative $f'(z)$ has there a zero of order at least two, and consequently if $u(x, y)$ is harmonic at infinity its first and second order partial derivatives vanish there; if $f'(z)$ has a zero there of order k (> 2), then both $f(z)$ and $u(x, y)$ have critical points there, of order $k - 2$, and conversely. A finite or infinite critical point z_0 of an analytic or harmonic function of order m retains the property of being a critical point of order m under one-to-one conformal transformation of the neighborhood of z_0, even if a finite z_0 is transformed to infinity or an infinite z_0 is transformed to a finite point.

If a sequence of functions $u_k(x, y)$ harmonic in a region R converges uniformly in R to the function $u(x, y)$ harmonic but not identically constant in R, if (x_0, y_0) is a finite or infinite critical point of $u(x, y)$ in R of order m, and if $N(x_0, y_0)$ is a neighborhood of (x_0, y_0) whose closure lies in R and contains no other critical point of $u(x, y)$, then for k sufficiently large precisely m critical points of $u_k(x, y)$ lie in $N(x_0, y_0)$, each critical point being counted according to its multiplicity. The proof may be conveniently given by means of Hurwitz's Theorem, for if the functions $v(x, y)$ and $v_k(x, y)$ conjugate to $u(x, y)$ and $u_k(x, y)$ are suitably chosen, the functions $u_k(x, y) + iv_k(x, y)$ are analytic in $N(x_0, y_0)$ and converge uniformly there to the function $u(x, y) + iv(x, y)$.

We state for reference without proof (which may be given by inequalities derived from Poisson's integral)

HARNACK'S THEOREM. *Let $u_n(x, y)$ be a monotonically increasing sequence of functions harmonic in a region R. Then either $u_n(x, y)$ becomes infinite at every point of R or the sequence converges throughout R, uniformly on any closed set interior to R.*

§1.2. Gauss's Theorem. We commence our study of the location of critical points by considering the simplest non-trivial functions, namely polynomials. Rolle's Theorem of course applies to real polynomials as to any real function possessing a derivative, and informs us that between two zeros of the function lies at least one zero of the derivative. Beyond Rolle's Theorem, the first general result concerning the zeros of the derivative of an arbitrary polynomial seems to be due to Gauss [1816]:

GAUSS'S THEOREM. *Let $p(z)$ be the polynomial $(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$, and let a field of force be defined by fixed particles situated at the points $\alpha_1, \alpha_2, \cdots, \alpha_n$, where each particle repels with a force equal to the inverse distance. At a multiple zero of $p(z)$ are to be placed a number of particles equal to the multiplicity. Then the zeros of the derivative $p'(z)$ are, in addition to the multiple zeros of $p(z)$, precisely the positions of equilibrium in this field of force.*

The logarithmic derivative of $p(z)$ is

$$(1) \qquad \frac{p'(z)}{p(z)} = \frac{1}{z - \alpha_1} + \frac{1}{z - \alpha_2} + \cdots + \frac{1}{z - \alpha_n}.$$

The conjugate of $1/(z - \alpha_k)$ is a vector whose direction (including sense) is the direction from α_k to z and whose magnitude is the reciprocal of the distance from α_k to z, so this vector represents the force at the variable point z due to a single fixed particle at α_k. Every multiple zero (but no simple zero) of $p(z)$ is a zero of $p'(z)$; every other zero of $p'(z)$ is by (1) a position of equilibrium in the field of force; every position of equilibrium is by (1) a zero of $p'(z)$. Gauss's Theorem is established.

As a simple example here, we choose $p(z) = (z - \alpha_1)^{m_1}(z - \alpha_2)^{m_2}$, $m_1 m_2 \neq 0$. Except for α_1 and α_2, the only zero of $p'(z)$ is given by

$$\frac{m_1}{z - \alpha_1} + \frac{m_2}{z - \alpha_2} = 0, \qquad \frac{z - \alpha_1}{\alpha_2 - z} = \frac{m_1}{m_2},$$

so z is the point $(m_2\alpha_1 + m_1\alpha_2)/(m_1 + m_2)$ which divides the segment $\alpha_1\alpha_2$ in the ratio $m_1 : m_2$.

The particles of Gauss's Theorem may be termed of *unit mass*; more generally we may consider particles of arbitrary (not necessarily integral) positive mass, repelling with a force equal to the quotient of the mass by the distance. Thus we have the

COROLLARY. *If particles of positive masses $\mu_1, \mu_2, \cdots, \mu_n$ are placed at the respective points $\alpha_1, \alpha_2, \cdots, \alpha_n$, then the positions of equilibrium in the resulting field of force are precisely the zeros of the derivative of $p(z) = \prod_{k=1}^{n} (z - \alpha_k)^{\mu_k}$, except that α_k is also a zero of $p'(z)$ if we have $\mu_k > 1$.*

Gauss's Theorem is of such central importance in the sequel that if a polynomial $p(z)$ is given we often automatically set up the corresponding field of force, and identify a zero of $p(z)$ with a fixed particle.

§1.3. Lucas's Theorem. If all the zeros of a polynomial $p(z)$ are real, it follows from Rolle's Theorem that each interval J of the axis of reals bounded by successive zeros of $p(z)$ contains at least one zero of the derivative $p'(z)$. If the respective multiplicities of the zeros of $p(z)$ are $m_1, m_2, \cdots, m_k$, whose sum is n, the corresponding multiplicities of the same points as zeros of $p'(z)$ are the positive numbers among the set $m_1 - 1, m_2 - 1, \cdots, m_k - 1$, whose sum is $n - k$. There are $k - 1$ intervals J each containing in its interior at least one zero of $p'(z)$, which has a totality of $n - 1$ zeros, so each interval J contains in its interior precisely one zero of $p'(z)$. All zeros of $p'(z)$ lie in the smallest interval of the axis of reals which contains the zeros of $p(z)$. This consequence of Rolle's Theorem can be generalized to apply to an arbitrary polynomial.

§1.3.1. Statement and proof.

LUCAS'S THEOREM. *Let $p(z)$ be a polynomial* whose zeros are $\alpha_1, \alpha_2, \cdots, \alpha_n$, and let Π be the smallest convex set on which those zeros lie. Then all zeros of the derivative $p'(z)$ also lie on Π. No zero of $p'(z)$ lies on the boundary of Π unless it is a multiple zero of $p(z)$, or unless all the zeros of $p(z)$ are collinear.*

Let pegs be placed in the z-plane at the points $\alpha_1, \alpha_2, \cdots, \alpha_n$; let a large rubber band be stretched in the plane so as to to include all the pegs. If the rubber band is allowed to contract so as to rest only on the pegs, and if it remains taut, it will fit over the pegs in the form of the boundary of Π. Thus if $p(z)$ has but one distinct zero, Π coincides with that zero; if the zeros of $p(z)$ are collinear, Π is a line segment; in any other case Π is the closed interior of a rectilinear polygon, called the *Lucas polygon* for $p(z)$ or for the points α_j. In every case the set Π

* Here and in corresponding places throughout the present work, the qualifying adjective *non-constant* is tacitly understood.

is convex in the sense that if two points belong to Π, so also does the line segment joining them.

The set Π can also be defined as the point set common to all closed half-planes each containing all the α_k.

If z_0 is a point exterior to Π, there exists a line L separating z_0 and Π; in fact, L may be chosen as the perpendicular bisector of the shortest segment joining z_0 to Π. In the field of force set up by Gauss's Theorem, all particles lie on one side of L; the force at z_0 due to each particle has a non-vanishing component perpendicular to L directed toward the side of L on which z_0 lies. Consequently z_0 cannot be a position of equilibrium. Moreover z_0 cannot be a multiple zero of $p(z)$, hence cannot be a zero of $p'(z)$.

If z_0 is a boundary point of Π not a zero of $p(z)$, and if not all the zeros of $p(z)$ are collinear, denote by L the line containing the side of the boundary of Π on which z_0 lies. Then one of the two half-planes bounded by L contains points

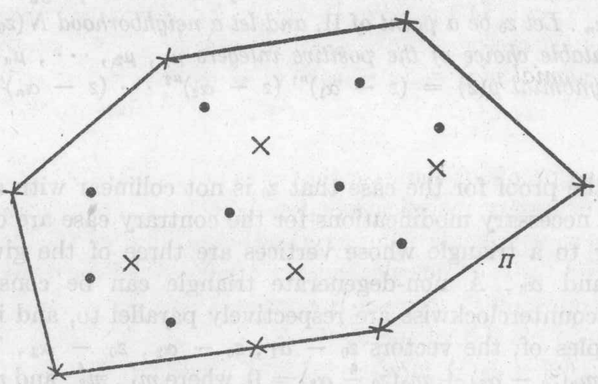

Fig. 1 illustrates §1.3.1 Lucas's Theorem

α_k in its interior while the other half-plane does not. Again the force at z_0 due to all the particles has a non-vanishing component perpendicular to L, so z_0 cannot be a position of equilibrium nor a zero of $p'(z)$. The proof is complete.

Although Lucas's Theorem follows at once from that of Gauss, there seems to be no evidence that it was stated by Gauss. It is credited to F. Lucas [1874]. Since his proof, as D. R. Curtiss [1922] has remarked, it has been "discovered and rediscovered, proved and reproved in most of the languages of Europe—and all the proofs are substantially the same".

In the present work we shall find numerous analogs and generalizations of Lucas's Theorem, many of them proved by this same method of considering at a point outside of a certain locus the total force in the field set up by Gauss's Theorem, and showing that this total force cannot be zero. In particular we shall make frequent use of the fact that a point z_0 cannot be a position of equilibrium under the action of several forces if those forces are represented by vectors having their initial points in z_0 and either their terminal points in an angle less than π

with vertex z_0 or their terminal points in an angle π with vertex z_0 and at least one terminal point interior to that angle.

§1.3.2. Complements. If the positions of the zeros of a variable polynomial $p(z)$ are fixed, and if the multiplicities of those zeros are permitted to take all possible values, the totality of critical points of $p(z)$ forms a countable point set S. Lucas's Theorem utilizes the positions of the zeros of a given polynomial but not their multiplicities, and essentially asserts that S lies in Π. However, we now prove that Π *is the closure of* S; in other words, if we wish to assign only a *closed* set as one in which the critical points lie, Lucas's Theorem is the best possible result [Fejér, Toeplitz, 1925; the corresponding result for real polynomials with non-real zeros (slightly weaker than §1.4.2 Theorem 1) had been previously proved (1920a) by this same method by the present writer]:

THEOREM 1. *Let Π be the closed interior of the Lucas polygon for the points $\alpha_1, \alpha_2, \cdots, \alpha_n$. Let z_0 be a point of Π, and let a neighborhood $N(z_0)$ of z_0 be given. Then for a suitable choice of the positive integers $\mu_1, \mu_2, \cdots, \mu_n$, the derivative $p'(z)$ of the polynomial $p(z) = (z - \alpha_1)^{\mu_1} (z - \alpha_2)^{\mu_2} \cdots (z - \alpha_n)^{\mu_n}$ has a zero in $N(z_0)$.*

We phrase the proof for the case that z_0 is not collinear with any two of the points α_k; the necessary modifications for the contrary case are obvious.* Then z_0 lies interior to a triangle whose vertices are three of the given points α_k, say α_1, α_2, and α_3. A non-degenerate triangle can be constructed whose sides directed counterclockwise are respectively parallel to, and indeed suitable positive multiples of, the vectors $z_0 - \alpha_1$, $z_0 - \alpha_2$, $z_0 - \alpha_3$. Thus we have $m_1'(z_0 - \alpha_1) + m_2'(z_0 - \alpha_2) + m_3'(z_0 - \alpha_3) = 0$, where m_1', m_2', and m_3' are positive, whence with $m_k' = m_k/(z_0 - \alpha_k)(\bar{z}_0 - \bar{\alpha}_k)$

$$\frac{m_1}{\bar{z}_0 - \bar{\alpha}_1} + \frac{m_2}{\bar{z}_0 - \bar{\alpha}_2} + \frac{m_3}{\bar{z}_0 - \bar{\alpha}_3} = 0,$$

where m_1, m_2, and m_3 are positive, If the positive rational numbers r_1, r_2, r_3 approach respectively m_1, m_2, m_3, and the positive rational numbers $r_4, r_5, \cdots, r_n$ approach zero, then a zero z of the function

$$\text{(1)} \qquad \frac{r_1}{z - \alpha_1} + \frac{r_2}{z - \alpha_2} + \cdots + \frac{r_n}{z - \alpha_n}$$

approaches z_0, and hence for suitable choice of $r_1, r_2, \cdots, r_n$ this zero of (1) lies in $N(z_0)$. If m is the least common multiple of the denominators of the latter numbers r_k, we need merely set $\mu_k = mr_k$ to complete the proof.

* It is not sufficient to assume here merely that z_0 is an interior point of Π; for instance if $p(z)$ is the polynomial $z^4 - 1$, the point $z = 0$ is interior to Π but not interior to a triangle whose vertices are zeros of $p(z)$.

The Corollary to Gauss's Theorem (§1.2) yields an extension of Lucas's Theorem:

COROLLARY. *If* Π *is the closed interior of the Lucas polygon for the points* $\alpha_1, \alpha_2, \cdots, \alpha_n$, *then* Π *contains the zeros of the derivative of the function*

$$(z - \alpha_1)^{\mu_1}(z - \alpha_2)^{\mu_2} \cdots (z - \alpha_n)^{\mu_n},$$

where all the μ_k *are positive. No zero of the derivative lies on the boundary of* Π *unless it is a point* α_k *with* $\mu_k > 1$, *or unless all the points* α_k *are collinear.*

Another general property of the zeros of the derivative of a polynomial is expressed in

THEOREM 2. *The zeros of a polynomial* $p(z)$ *of degree greater than unity and the zeros of its derivative* $p'(z)$ *have the same center of gravity.*

The center of gravity of the zeros of

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

is $-a_1/n$, as is the center of gravity of the zeros of

$$p'(z) = nz^{n-1} + (n-1)a_1 z^{n-2} + \cdots + a_{n-1}.$$

The significance of Theorem 2 lies especially in the fact that any line L through the center of gravity of a finite set of points bounds two closed half-planes each containing one or more of those points; either all the points lie on L, or at least one point lies interior to each of the half-planes.

Under the conditions of Lucas's Theorem, the center of gravity of the α_k lies in Π, and is an interior point of Π unless all the α_k are collinear. Any point z_0 of Π has the property that any line through z_0 bounds two closed half-planes each containing one or more of the α_k; this property is shared by no point exterior to Π. If z_0 is an interior point of Π, then at least one α_k lies on each side of every line through z_0.

§1.4. Jensen's Theorem. For a real polynomial $p(z)$, non-real zeros occur in pairs of conjugate imaginaries; this symmetry may enable us to improve Lucas's Theorem as applied to $p(z)$. Using as diameter the segment joining each pair of conjugate imaginary zeros of $p(z)$, we construct a circle, which we shall call a *Jensen circle*. We shall prove [Jensen, 1913; proof due to Walsh, 1920a]:

JENSEN'S THEOREM. *Each non-real zero of the derivative* $p'(z)$ *of a real polynomial* $p(z)$ *lies on or within a Jensen circle for* $p(z)$. *A non-real point* z_0 *not a multiple zero of* $p(z)$ *nor interior to a Jensen circle for* $p(z)$ *is a zero of* $p'(z)$ *only if* $p(z)$ *has no real zeros and* z_0 *lies on all Jensen circles.*