INTRODUCTION TO QUADRATIC FORMS

BY

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Preface

The main purpose of this book is to give an account of the fractional and integral classification problem in the theory of quadratic forms over the local and global fields of algebraic number theory. The first book to investigate this subject in this generality and in the modern setting of geometric algebra is the highly original work *Quadratische Formen und orthogonale Gruppen* (Berlin, 1952) by M. EICHLER. The subject has made rapid strides since the appearance of this work ten years ago and during this time new concepts have been introduced, new techniques have been developed, new theorems have been proved, and new and simpler proofs have been found. There is therefore a need for a systematic account of the theory that incorporates the developments of the last decade.

The classification of quadratic forms depends very strongly on the nature of the underlying domain of coefficients. The domains that are really of interest are the domains of number theory: algebraic number fields, algebraic function fields in one variable over finite constant fields, all completions thereof, and rings of integers contained therein. Part One introduces these domains via valuation theory. The number theoretic and function theoretic cases are handled in a unified way using the Product Formula, and the theory is developed up to the Dirichlet Unit Theorem and the finiteness of class number. It is hoped that this will be of service, not only to the reader who is interested in quadratic forms, but also to the reader who wishes to go deeper into algebraic number theory and class field theory. In Part Two there is a discussion of topics from abstract algebra and geometric algebra which will be used later in the arithmetic theory. Part Three treats the theory of quadratic forms over local and global fields. The direct use of local class field theory has been circumvented by introducing the concept of the quadratic defect (which is needed later for the integral theory) right at the start. The quadratic defect gives, in effect, a systematic way of refining certain types of quadratic approximations. However, the global theory of quadratic forms does present a dilemma. Global class field theory is still so inaccessible that it is not possible merely to quote results from the literature. On the other hand a thorough development of global class field theory cannot be included in a book of this size and scope. We have therefore decided to compromise by specializing the methods of global class field theory to the case of quadratic extensions, thereby obtaining all that is needed for the global theory of quadratic forms. Part Four starts with a systematic development of the formal aspects of integral quadratic forms over Dedekind domains. These techniques are then applied, first to solve the local integral classification problem, then to investigate the global integral theory, in particular to establish the relation between the class, the genus, and the spinor genus of a quadratic form.

It must be emphasized that only a small part of the theory of quadratic forms is covered in this book. For the sake of simplicity we confine ourselves entirely to quadratic forms and the orthogonal group, and then to a particular part of this theory, namely to the classification problem over arithmetic fields and rings. Thus we do not even touch upon the theory of hermitian forms, reduction theory and the theory of minima, composition theory, analytic theory, etc. For a discussion of these matters the reader is referred to the books and articles listed in the bibliography.

O. T. O'Meara

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O. T. O'Meara

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[1] Alfred P. Sloan Fellow, 1960—1963.

Prerequisites and Notation

If X and Y are any two sets, then $X \subset Y$ will denote strict inclusion, $X - Y$ will denote the difference set, $X \rightarrow Y$ will denote a surjection of X onto Y, $X \rightarrowtail Y$ an injection, $X \rightarrowtail\!\!\!\rightarrow Y$ a bijection, and $X \nrightarrow Y$ an arbitrary mapping. By "almost all elements of X" we shall mean "all but a finite number of elements of X".

N denotes the set of natural numbers, **Z** the set of rational integers, **Q** the set of rational numbers, **R** the set of real numbers, **P** the set of positive numbers, and **C** the set of complex numbers.

We assume a knowledge of the elementary definitions and facts of general topology, such as the concepts of continuity, compactness, completeness and the product topology.

From algebra we assume a knowledge of 1) the elements of group theory and also the fundamental theorem of abelian groups, 2) galois theory up to the fundamental theorem and including the description of finite fields, 3) the rudiments of linear algebra, 4) basic definitions about modules.

If X is any additive group, in particular if X is either a field or a vector space, then $\dot{X}$ will denote the set of non-zero elements of X. If H is a subgroup of a group G, then $(G:H)$ is the index of H in G. If E/F is an extension of fields, then $[E:F]$ is the degree of the extension. The characteristic of F will be written $\chi(F)$. If α is an element of E that is algebraic over F, then $\operatorname{irr}(x, \alpha, F)$ is the irreducible monic polynomial in the variable x that is satisfied by α over the field F. If E_1 and E_2 are subfields of E, then E_1E_2 denotes the compositum of E_1 and E_2 in E. If E/F is finite, then $N_{E/F}$ will denote the norm mapping from E to F; and $S_{E/F}$ will be the trace.

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Contents

Part Three

Arithmetic theory of quadratic forms over fields

Part Four

Arithmetic theory of quadratic forms over rings

Part One

Arithmetic Theory of Fields

Chapter I

Valuated Fields

The descriptive language of general topology is known to all mathematicians. The concept of a valuation allows one to introduce this language into the theory of algebraic numbers in a natural and fruitful way. We therefore propose to study some of the connections between valuation theory, algebraic number theory, and topology. Strictly speaking the topological considerations are just of a conceptual nature and in fact only the most elementary results on metric spaces and topological groups will be used; nevertheless these considerations are essential to the point of view taken throughout this chapter and indeed throughout the entire book.

§ 11. Valuations

§ 11 A. The definitions

Let F be a field. A valuation on F is a mapping $|\ |$ of F into the real numbers R which satisfies

$$(V_1)\quad |\alpha| > 0 \quad \text{if} \quad \alpha \neq 0, \quad |0| = 0$$

$$(V_2)\quad |\alpha\beta| = |\alpha| \cdot |\beta|$$

$$(V_3)\quad |\alpha + \beta| \leqq |\alpha| + |\beta|$$

for all α, β in F. A mapping which satisfies (V_1), (V_2) and

$$(V_{3'})\quad |\alpha + \beta| \leqq \max(|\alpha|, |\beta|)$$

will satisfy (V_3) and will therefore be a valuation. Axiom (V_3) is called the triangle law, axiom $(V_{3'})$ is called the strong triangle law. A valuation which satisfies the strong triangle law is called non-archimedean, a valuation which does not satisfy the strong triangle law is called archimedean. Non-archimedean valuations will be used to describe certain properties of divisibility in algebraic number theory.

The mapping $\alpha \to |\alpha|$ is a multiplicative homomorphism of $\dot{F}$ into the positive real numbers, and so the set of images of $\dot{F}$ forms a multiplicative subgroup of R. We call the set

$$|F| = \{|\alpha| \in \mathsf{R} \mid \alpha \in \dot{F}\}$$

the value group of F under the given valuation. We have the equations

$$|1| = 1, \quad |-\alpha| = |\alpha|, \quad |\alpha^{-1}| = |\alpha|^{-1},$$

and also

$$\big|\, |\alpha| - |\beta| \,\big| \leqq |\alpha - \beta| .$$

Every field F has at least one valuation, the trivial valuation obtained by putting $|\alpha| = 1$ for all α in $\dot{F}$. Such a valuation satisfies the strong triangle law and is therefore non-archimedean. A finite field can possess only the trivial valuation since, if we let q stand for the number of elements in F, we have

$$|\alpha|^{q-1} = |\alpha^{q-1}| = |1| = 1 \qquad \forall \alpha \in \dot{F} .$$

Any subfield F of the complex numbers C can be regarded as a valuated field by restricting the ordinary absolute value from C to F. Conversely, it will follow from the results of § 12 that every field with an archimedean valuation is obtained essentially in this way. A valuated field which contains the rational numbers Q and which induces the ordinary absolute value on Q must be archimedean since $|1 + 1| = 2 > 1$.

Now a few words about the topological properties of the valuated field F. First we notice that F can be regarded as a metric topological space in a natural way: define the distance between two points α and β of F to be $|\alpha - \beta|$. If we take this topology on F and the product topology on $F \times F$, then it is easily seen by elementary methods that the mappings

$$(\alpha, \beta) \to \alpha + \beta \quad \text{and} \quad (\alpha, \beta) \to \alpha\beta$$

of $F \times F$ into F are continuous. So are the mappings

$$\alpha \to -\alpha \quad \text{and} \quad \alpha \to \alpha^{-1}$$

of F into F and of $\dot{F}$ into $\dot{F}$, respectively. These four facts simply mean, in the language of topological groups, that F is a topological field. Hence the mappings

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) \to \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

and

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) \to \alpha_1 \alpha_2 \ldots \alpha_n$$

of $F \times \cdots \times F$ into F are continuous. Hence a polynomial with coefficients in F determines a continuous function of $F \times \cdots \times F$ into F; and a rational function is continuous at any point of $F \times \cdots \times F$ at which its denominator is not zero. The inequality $||\alpha| - |\alpha_0|| \leqq |\alpha - \alpha_0|$ shows that the mapping

$$\alpha \to |\alpha|$$

of F into $\mathbf{R}$ is continuous.

The limit of a sequence and the sum of a series can be defined as it is usually defined in a first course on the calculus. We find that if $\alpha_n \to \alpha$ and $\beta_n \to \beta$ as $n \to \infty$, then

$$\alpha_n \pm \beta_n \to \alpha \pm \beta, \quad \alpha_n \beta_n \to \alpha \beta$$

$$\alpha_n^{-1} \to \alpha^{-1} \quad \text{if} \quad \alpha \neq 0$$

$$|\alpha_n| \to |\alpha| .$$

Similarly if $\sum_1^\infty a_\lambda$ and $\sum_1^\infty b_\lambda$ converge, then so do

$$\sum_1^\infty (a_\lambda \pm b_\lambda) = \sum_1^\infty a_\lambda \pm \sum_1^\infty b_\lambda .$$

The terms of any convergent series must tend to 0.

The closure $\hat{G}$ of a subfield G of F is again a subfield of F. For we can find $\alpha_n \to \alpha$ and $\beta_n \to \beta$ with α_n and β_n in G whenever α and β are given in $\hat{G}$; then

$$\alpha + \beta = \lim (\alpha_n + \beta_n) \in \hat{G} .$$

Hence $\hat{G}$ is closed under addition. Similarly with multiplication and inversion. Hence $\hat{G}$ is a field.

Closely related to the concept of a valuation is the concept of an analytic map. An analytic map is an isomorphism φ of the valuated field F onto a valuated field F' such that $|\varphi\alpha| = |\alpha|$ holds for all α in F. In other words, an analytic map preserves the valuation as well as the algebraic structure. An analytic map φ is therefore a topological isomorphism between the topological fields F and F'. Suppose now that F' is just any abstract field, but that F is still the valuated field under discussion. Also suppose that we have an isomorphism φ of F onto F'. We can then define a valuation on F' by putting $|\beta| = |\varphi^{-1}\beta|$ for all β in F'. When we perform this construction we shall say that φ has carried the valuation from F to F'. Clearly the valuation just defined makes φ analytic.

We conclude this subparagraph with an important example. Consider the rational numbers $\mathbf{Q}$ and a fixed prime number p. A typical $\alpha \in \dot{\mathbf{Q}}$ can be written in the form

$$\alpha = p^i \left(\frac{m}{n}\right)$$

with m and n prime to p. Do this with each α and put

$$|\alpha|_p = \left(\frac{1}{p}\right)^i.$$

It is easy to show that this defines a non-archimedean valuation on $\mathbf{Q}$. To say that α is small under this valuation means that it is highly divisible by p. (Here is our first glimpse of the connection between valuations and number theory; we shall return to this example in a more general setting in Chapter III.)

§ 11B. Non-archimedean valuations

11:1. *A valuation on a field F is non-archimedean if and only if it is bounded on the natural integers of F.*

Proof. We recall that the natural integers in an arbitrary field F are the finite sums of the form $1 + \cdots + 1$. We need only do the sufficiency. Thus we are given a fixed positive bound M such that $|m| \leqq M$ holds for any natural integer m in F. Then

$$\begin{aligned} |\alpha + \beta|^n &= |(\alpha + \beta)^n| \\ &= |\alpha^n + \tbinom{n}{1} \alpha^{n-1} \beta + \cdots + \beta^n| \\ &\leqq |1|\,|\alpha|^n + |\tbinom{n}{1}|\,|\alpha|^{n-1}\,|\beta| + \cdots + |1|\,|\beta|^n \\ &\leqq M\,\{|\alpha|^n + |\alpha|^{n-1}\,|\beta| + \cdots + |\beta|^n\} \\ &\leqq M\,(n+1)\,\{\max(|\alpha|, |\beta|)\}^n. \end{aligned}$$

Hence

$$|\alpha + \beta| \leqq M^{1/n} (n+1)^{1/n} \max(|\alpha|, |\beta|)\,.$$

If we let $n \to \infty$ we obtain the result. **q. e. d.**

This result has two immediate consequences. First, a field of characteristic $p > 0$ can have no archimedean valuations. Second, a valuated extension field E of F is non-archimedean if and only if F is non-archimedean under the induced valuation.

11:2. **Principle of Domination.** *In a non-archimedean field we have*

$$|\alpha_1 + \cdots + \alpha_n| = |\alpha_1|$$

if $|\alpha_\lambda| < |\alpha_1|$ for $\lambda > 1$.

Proof. It suffices to prove $|\alpha + \beta| = |\alpha|$ when $|\alpha| > |\beta|$. We have

$$|\alpha| = |-\beta + \alpha + \beta| \leqq \max(|\beta|\,, |\alpha + \beta|)\,,$$

and so $|\alpha| \leqq |\alpha + \beta|$. But

$$|\alpha + \beta| \leqq \max(|\alpha|\,, |\beta|) = |\alpha|\,.$$

Hence $|\alpha + \beta| = |\alpha|$. **q. e. d.**

11:2a. *Suppose that* $\sum_1^\infty \alpha_\lambda$ *is convergent. If* $|\alpha_\lambda| < |\alpha_1|$ *for* $\lambda > 1$, *then*

$$|\sum_1^\infty \alpha_\lambda| = |\alpha_1| .$$

11:3. *Let E/F be an algebraic extension of fields. Suppose a valuation on E induces the trivial valuation on F. Then the valuation is trivial on E.*

Proof. For suppose that the given valuation is non-trivial on E. Then we can find $\alpha \in E$ with $|\alpha| > 1$. Let us write

$$\alpha^n + a_1 \alpha^{n-1} + \cdots + a_{n-1} \alpha + a_n = 0$$

with all a_i in F. Now all $|a_i|$ are either 0 or 1 since the valuation is trivial on F. And $|\alpha^n| > |\alpha^i|$ whenever $n > i$. Hence

$$|\alpha^n| = |\alpha^n + a_1 \alpha^{n-1} + \cdots + a_n|$$

by the Principle of Domination. Hence $|\alpha^n| = 0$, and this is absurd. q. e. d.

We shall see later in Chapter III that the above result does not hold if the extension E/F is transcendental.

§ 11 C. Equivalent valuations

Consider two valuations $|\ |_1$ and $|\ |_2$ on the same field F. We say that $|\ |_1$ and $|\ |_2$ are equivalent valuations if they define the same topology on F. It is clear that equivalence of valuations is an equivalence relation on the set of all valuations on F.

11:4. *Let* $|\ |_1$ *and* $|\ |_2$ *be two valuations on the same field F. Then the following assertions are equivalent:*

(1) *The two valuations are equivalent,*

(2) $|\alpha|_1 < 1 \Leftrightarrow |\alpha|_2 < 1$,

(3) *There is a positive number* ϱ *such that* $|\alpha|_1^\varrho = |\alpha|_2$ *for all* α *in F.*

Proof. (1) $\Rightarrow$ (2). On grounds of symmetry it is enough to consider an α in F with $|\alpha|_1 < 1$ and to prove that $|\alpha|_2 < 1$. Let

$$N = \{x \in F \mid |x|_2 < 1\} .$$

This set N is a neighborhood of 0 under the topology induced by either valuation. Now $|\alpha^n|_1 = |\alpha|_1^n$ can be made arbitrarily small by choosing n large enough, in particular we can choose an n such that $\alpha^n \in N$. But then $|\alpha|_2^n < 1$. And so $|\alpha|_2 < 1$.

(2) $\Rightarrow$ (3). By taking inverses we deduce from (2) that $|\alpha|_1 > 1$ if and only if $|\alpha|_2 > 1$. Hence $|\alpha|_1 = 1$ if and only if $|\alpha|_2 = 1$. In particular, if one of the valuations is trivial then so is the other. We may therefore assume that neither valuation is trivial.

Take α_0 in F with $0 < |\alpha_0|_2 < 1$. Then $0 < |\alpha_0|_1 < 1$ by hypothesis. Hence we have $|\alpha_0|_2 = |\alpha_0|_1^\varrho$ where

$$\varrho = \log |\alpha_0|_2 / \log |\alpha_0|_1 > 0 .$$

We claim that $|\alpha|_2 = |\alpha|_1^\varrho$ for all α in F. For suppose if possible that there is an α for which $|\alpha|_2$ and $|\alpha|_1^\varrho$ are not equal. Replacing α by its inverse if necessary allows us to assume that $|\alpha|_2 < |\alpha|_1^\varrho$. Now choose a rational number m/n with $n > 0$ such that

$$|\alpha|_2 < |\alpha_0|_2^{m/n} = |\alpha_0|_1^{\varrho m/n} < |\alpha|_1^\varrho .$$

This gives

$$|\alpha^n/\alpha_0^m|_2 < 1 \quad \text{and} \quad |\alpha^n/\alpha_0^m|_1 > 1$$

which denies our hypothesis. Hence our supposition about α is false. Hence (3) follows.

(3) $\Rightarrow$ (1). This part is clear. q. e. d.

11:4a. *Suppose* $|\ |_1$ *is non-trivial. Then* $|\ |_1$ *is equivalent to* $|\ |_2$ *if*

$$|\alpha|_1 < 1 \Rightarrow |\alpha|_2 < 1 .$$

Proof. If $|\alpha|_1 > 1$, then $|\alpha|_2 > 1$ by taking inverses. It is therefore enough to prove

$$|\alpha|_1 = 1 \Rightarrow |\alpha|_2 = 1 .$$

Choose $\beta \in F$ with $0 < |\beta|_1 < 1$. Then

$$|\alpha^n \beta|_1 < 1 \Rightarrow |\alpha^n \beta|_2 < 1 \Rightarrow |\alpha|_2^n\, |\beta|_2 < 1 .$$

It follows from the last inequality, by letting $n \to \infty$, that $|\alpha|_2 \leqq 1$. Replace α by α^{-1}. This gives $|\alpha|_2 \geqq 1$. Hence $|\alpha|_2 = 1$. q. e. d.

11:4b. *The trivial valuation is equivalent to itself and itself alone.*

11:5. *Let* $|\ |_1$ *and* $|\ |_2$ *be two equivalent valuations on a field E and let F be an arbitrary subfield of E. Suppose the two valuations induce the same non-trivial valuation on F. Then* $|\ |_1$ *and* $|\ |_2$ *are equal on E.*

Proof. We have a positive number ϱ such that $|\alpha|_1^\varrho = |\alpha|_2$ for all α in E. Choose $\alpha_0 \in F$ with

$$0 < |\alpha_0|_1 = |\alpha_0|_2 < 1 .$$

Then $|\alpha_0|_1^\varrho = |\alpha_0|_1$. Hence $\varrho = 1$. q. e. d.

Consider the valuation $|\ |$ on our field F and let ϱ be any positive number. We know that $|\ |^\varrho$, if it is a valuation, will be equivalent to $|\ |$. Of course $|\ |^\varrho$ need not be a valuation at all; for instance the ordinary absolute value on $\mathbf{Q}$ with $\varrho > 1$ gives

$$|1 + 1|^\varrho = 2^\varrho > 2 = |1|^\varrho + |1|^\varrho .$$

However $|\ |^\varrho$ is a valuation whenever $0 < \varrho \leqq 1$. To see this we observe that $|\alpha + \beta|^\varrho \leqq (|\alpha| + |\beta|)^\varrho$; it therefore suffices to prove that

$$(|\alpha| + |\beta|)^\varrho \leqq |\alpha|^\varrho + |\beta|^\varrho.$$

But

$$1 = \frac{|\alpha|}{|\alpha| + |\beta|} + \frac{|\beta|}{|\alpha| + |\beta|} \leq \left(\frac{|\alpha|}{|\alpha| + |\beta|}\right)^{\varrho} + \left(\frac{|\beta|}{|\alpha| + |\beta|}\right)^{\varrho},$$

since $0 < \varrho \leqq 1$. So it is true.

In the non-archimedean case things are simpler. The strong triangle law must obviously hold for $|\ |^{\varrho}$ if it holds for $|\ |$, even if ϱ is greater than 1. Hence $|\ |^{\varrho}$ is a valuation if $|\ |$ is non-archimedean and $\varrho > 0$. It is clear that $|\ |^{\varrho}$ is non-archimedean if and only if $|\ |$ is.

§ 11D. Prime spots

Consider a field F. By a prime spot, or simply a spot, on F we mean a single class of equivalent valuations on F; thus a spot is a certain set of maps of F into $\mathbf{R}$. Consider a prime spot $\mathfrak{p}$ on F. Each valuation $|\ |_{\mathfrak{p}} \in \mathfrak{p}$ defines the same topology on F by the definition of a prime spot. We call this the $\mathfrak{p}$-adic topology on F. If $\mathfrak{p}$ contains the trivial valuation (in which case it can contain no other) we call $\mathfrak{p}$ the trivial spot on F. In the same way we can define archimedean and non-archimedean spots. If $\mathfrak{p}$ is non-trivial it will contain an infinite number of valuations. Two spots on F are equal if and only if their topologies are the same.

Suppose $\sigma: F \rightarrowtail F'$ is an isomorphism of a field F with a spot $\mathfrak{p}$ onto an abstract field F'. It is easily seen that *there is a unique spot* $\mathfrak{q}$ *on* F' *which makes* σ *topological:* the existence of $\mathfrak{q}$ is obtained by letting σ carry some valuation in $\mathfrak{p}$ over to F', and the uniqueness of $\mathfrak{q}$ follows from the fact that both σ and σ^{-1} will be topological. In this construction we say that σ carries the spot $\mathfrak{p}$ to F'. The unique spot on F' that makes σ topological will be written

$$\mathfrak{p}^{\sigma}.$$

To each $|\ |_{\mathfrak{p}} \in \mathfrak{p}$ there corresponds a valuation $|\ |_{\mathfrak{p}^{\sigma}} \in \mathfrak{p}^{\sigma}$ such that

$$|\beta|_{\mathfrak{p}^{\sigma}} = |\sigma^{-1}\beta|_{\mathfrak{p}} \quad \forall\, \beta \in F',$$

namely the valuation obtained by carrying $|\ |_{\mathfrak{p}}$ to F'.

11:6. *Let F and G be two fields provided with* $\mathfrak{p}$*-adic and* $\mathfrak{q}$*-adic topologies respectively. Let σ be a topological isomorphism of F onto G. Then* $\mathfrak{q} = \mathfrak{p}^{\sigma}$. *And for each* $|\ |_{\mathfrak{p}} \in \mathfrak{p}$ *there is a* $|\ |_{\mathfrak{q}} \in \mathfrak{q}$ *which makes σ analytic.*

Proof. Clearly $\mathfrak{q} = \mathfrak{p}^{\sigma}$ by definition of $\mathfrak{p}^{\sigma}$. Then $|\ |_{\mathfrak{q}}$ is simply the valuation $|\ |_{\mathfrak{p}^{\sigma}}$ defined above. **q. e. d.**

Let $\mathfrak{P}$ be a prime spot on an extension E of F. Each valuation in $\mathfrak{P}$ induces a valuation on F, and all valuations of F that are obtained from $\mathfrak{P}$ in this way are equivalent. Hence $\mathfrak{P}$ determines a unique spot $\mathfrak{p}$ on F. We say that $\mathfrak{P}$ induces $\mathfrak{p}$, or that $\mathfrak{P}$ divides $\mathfrak{p}$, and we write

$$\mathfrak{P} | \mathfrak{p}.$$

Whenever we refer to the spot $\mathfrak{P}$ on F we shall really mean that spot $\mathfrak{p}$ on F which is divisible by $\mathfrak{P}$. Here the $\mathfrak{P}$-adic topology on E induces the $\mathfrak{p}$-adic topology on F. We refer to this induced topology as the $\mathfrak{P}$-adic topology on F.

Now consider a set of prime spots S on F and another set T on E. We say that T divides S and write $T|S$ if the spot induced on F by each spot $\mathfrak{P}$ in T is in S. It is clear that there is an absolutely largest set of spots T on E which divides a given set S on F; we then say that T fully divides S and we write $T\|S$. One often uses the same letter S to denote the set of spots on E which fully divides the given set S on F.

§ 11E. The Weak Approximation Theorem

11:7. *Let* $|\ |_\lambda$ $(1 \leqq \lambda \leqq n)$ *be a finite number of inequivalent non-trivial valuations on a field F. Then there is an* $\alpha \in F$ *such that* $|\alpha|_1 > 1$ *and* $|\alpha|_\lambda < 1$ *for* $2 \leqq \lambda \leqq n$.

Proof. If $n = 1$ it is simply the fact that $|\ |_1$ is non-trivial. Next let $n = 2$. Since $|\ |_1$ and $|\ |_2$ are inequivalent we can find b, c in F such that

$$|b|_1 < 1\,, \quad |b|_2 \geqq 1\,, \quad |c|_1 \geqq 1\,, \quad |c|_2 < 1\,.$$

Then $\alpha = c/b$ does the job.

We continue by induction to n. First choose b with $|b|_1 > 1$ and $|b|_\lambda < 1$ $(2 \leqq \lambda \leqq n-1)$, then c with $|c|_1 > 1$ and $|c|_n < 1$. If $|b|_n < 1$ we are through. If $|b|_n = 1$, form cb^r and observe that for sufficiently large values of r we have

$$|cb^r|_1 > 1\,,\ |cb^r|_\lambda < 1 \quad (2 \leqq \lambda \leqq n)\,;$$

take $\alpha = cb^r$. Finally consider $|b|_n > 1$. Using the fact that $1 + b^r \to 1$ if $|b| < 1$ we easily see that

$$\left|\frac{cb^r}{1+b^r}\right|_\lambda \to \begin{cases} |c|_\lambda & \text{if } \lambda = 1 \quad \text{or} \quad \lambda = n \\ 0 & \text{if } 2 \leqq \lambda \leqq n-1\,. \end{cases}$$

This time take $\alpha = cb^r/(1 + b^r)$ with a sufficiently large r. **q. e. d.**

11:8. **Theorem.** *Let* $|\ |_\lambda$ $(1 \leqq \lambda \leqq n)$ *be a finite number of inequivalent non-trivial valuations on a field F. Consider n field elements* α_λ $(1 \leqq \lambda \leqq n)$. *Then for each* $\varepsilon > 0$ *there is an* $\alpha \in F$ *such that* $|\alpha - \alpha_\lambda|_\lambda < \varepsilon$ *for* $1 \leqq \lambda \leqq n$.

Proof. For each i $(1 \leqq i \leqq n)$ we can find $b_i \in F$ such that $|b_i|_i > 1$ and $|b_i|_\lambda < 1$ when $\lambda \neq i$. If we let $r \to \infty$ we see that

$$\frac{b_i^r}{1+b_i^r} \to \begin{cases} 1 & \text{under} \quad |\ |_i \\ 0 & \text{under} \quad |\ |_\lambda \quad \text{if} \quad \lambda \neq i\,. \end{cases}$$

Hence

$$c_r = \sum_{i=1}^{n} \frac{\alpha_i b_i^r}{1+b_i^r} \to \alpha_i$$

under the topology defined by $| \ |_t$. Then $\alpha = c_r$ with a sufficiently large r is the α we require. **q. e. d.**

§ 11F. Complete valuations and complete spots

Consider the distance function $d(\alpha, \beta) = |\alpha - \beta|$ associated with the valuation $| \ |$ on F. We can follow the language of metric topology and introduce the concept of a Cauchy sequence and completeness with respect to $d(\alpha, \beta)$. Completeness of $| \ |$ then means, by definition, that every Cauchy sequence converges to a limit in F.

11:9. **Example.** We have already mentioned that the terms of any convergent series over a valuated field must tend to 0. If F is a field with a complete non-archimedean valuation there is the following remarkable converse: every infinite series whose terms tend to 0 is convergent. For if we form the partial sums $A_1, \ldots, A_n, \ldots$ of $\sum_1^\infty \alpha_\lambda$ we see from the strong triangle law that

$$|A_m - A_n| \leq \max(|a_{n+1}|, \ldots, |a_m|),$$

hence the partial sums form a Cauchy sequence, hence $\sum_1^\infty \alpha_\lambda$ has a limit in F.

Let $\mathfrak{p}$ be a spot on the field F. We say that F is complete at $\mathfrak{p}$, or simply that F is complete, if there is at least one complete valuation in $\mathfrak{p}$. Because of the formula $| \ |_1^\varrho = | \ |_2$ relating equivalent valuations we see that if F is complete at $\mathfrak{p}$, then every valuation in $\mathfrak{p}$ is complete.

By a completion of a field F at one of its spots $\mathfrak{p}$ we mean a composite object consisting of a field E and a prime spot $\mathfrak{P}$ on E with the following properties:

1. E is complete at $\mathfrak{P}$,
2. F is a subfield of E and $\mathfrak{P} | \mathfrak{p}$,
3. F is dense in E.

We shall often shorten the terminology and just refer to a completion E of a given field F; this will of course mean that we have a certain prime spot $\mathfrak{p}$ on F in mind and that E is really a composite object consisting of the field E and a prime spot $\mathfrak{P}$ on E.

11:10. **Example.** A complete field is its own completion. It has no other completion.

11:11. **Example.** Consider the trivial spot $\mathfrak{p}$ on F. Here every Cauchy sequence has the form

$$\alpha_1, \ldots, \alpha_n, \alpha, \ldots, \alpha, \ldots$$

and this converges to α. Hence F is complete.