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Generalized Metric Spaces and Mappings

Shou Lin Ziqiu Yun（林 寿 恽自求）

（广义度量空间与映射）

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Foreword

What are generalized metric spaces? Most often, this expression denotes a system of classes of topological spaces, each of which is defined with the help of some typical topological property of metrizable spaces. For example, the classes of paracompact spaces, Moore spaces, spaces with a point-countable base, submetrizable spaces, perfectly normal spaces, first-countable spaces belong to this system. We also include in it the class of symmetrizable spaces and the class of Δ-metrizable spaces. The topologies of these spaces are generated by generalized metrics. In this way we obtain a rich system of classes of topological spaces which are all emerging, growing and spreading in many directions, from the same powerful germ, - the concept of the topology generated by a metric.

The system of generalized metrizable spaces (below we often use an abbreviation "GMS-system" to denote it) also includes various relations between its members. The simplest among them is, of course, the inclusion relation. But much deeper, often unpredictable links between classes of topological spaces are established by means of mappings. Mappings of various kinds serve as instruments by means of which the abstract geometric objects, such as topological spaces, topological groups, metric spaces, function spaces, and so on, can be compared one to another and further classified. This can be done at the level of individual spaces, but similarly, natural classes of mappings can be used to establish fundamental connections between classes of topological spaces.

The ideas and problems of the theory of generalized metrizable spaces, in particular, the general metrization problem, greatly influenced all domains of set-theoretic topology. A goal of this book is to describe the present structure of the theory of generalized metric spaces and the modern tendencies in this theory. An important fact: this is the first book on generalized metrizable spaces. Some brilliant survey papers on this topic were written by prominent topologists R. Hodel, G. Gruenhage, D. Burke, D. Lutzer, H.H. Wicke and J. Worrell, W. Fleissner, G.M. Reed, who themselves contributed greatly to the field. But a book on this rapidly developing subject was lacking. It should have appeared at least twenty years earlier. This has happened in China, where a version of this book has appeared in Chinese in 1995. So it is not astonishing that a strong input in the set-theoretic topology was made by in the last twenty or thirty years by Guoshi Gao, Zhimin Gao, Shou Lin, Chuan Liu, Fucai Lin, Ziqiu Yun, Liang xue Peng, Lei Mou, Wei xue Shi, Li hong Xie, Jing

Zhang, and many others.

It is an excellent news that, finally, an expanded and polished international edition of this book will appear in English. The contents are well selected. The book has three chapters and two appendices. The main part, consisting of the three chapters, is written in a very systematic way. Theorems and examples form a part of the big picture, they are supplied with detailed proofs. Stratifiable spaces, spaces with a σ-discrete network, bases of countable order and their theory developed by H.H. Wicke and J. Worrell, uniform bases introduced by P.S. Alexandroff, monotonically normal spaces of R.W. Heath, p-spaces and their theory, quotients of separable metrizable spaces, A.H. Stone's theorem on paracompactness of metric spaces, are presented with a great force. All major theorems obtained in the last fifty years in GMS-theory, in particular, V.I. Ponomarev's theorem characterizing first-countable spaces as open continuous images of metric spaces, V.V. Filippov's theorem on metrizability of paracompact p-spaces with a point-countable base, generalizing A.S. Miščenko's theorem on compacta with such a base, E. Michael's theorem on preservation of paracompactness by closed continuous mappings, a characterization of perfect preimages of metrizable spaces as paracompact p-spaces (by A.V. Arhangel'skiĭ), another, independent, characterization of the same spaces by K. Morita (as paracompact M-spaces), all these basic results of the GMS-theory the reader will find in the book. Appendices A and B are written in the form of survey, they provide the reader with many further advanced and more special facts and open questions, with metamathematical discussions and attractive philosophical insights, omitting detailed proofs. Much of additional information on the history of the subject is also provided here. An attractive feature of this book is its international character. The list of references is very rich, it includes about 500 entries, and will be of great value for readers. Besides the papers of mathematicians we have already mentioned above, the list includes the classical ground-breaking papers on GMS-theory, written by R. Bing, C. Dowker, J. Nagata, E. Michael, M.E. Rudin, A.H. Stone, Yu.M. Smirnov, K. Nagami, Z. Frolík, R. Engelking, Z. Balogh, J. van Mill, M. Hušek, H. Tamano, P. Nyikos, J. Chaber, S. Nedev, C. Borges, T. Hoshina, R. Stephenson, Y. Tanaka, H. Junnila, N.V. Veličko, M. Itō, D. Shakhmatov, M. Sakai, Y. Yajima, and others.

The book, written by Professors Shou Lin and Ziqiu Yun who are active experts in the GMS-theory with a long list of original contributions, will be a most valuable source of information for graduate and undergraduate topology students wishing to start their own research in this field or to apply it to other mathematical disciplines. It will be also very helpful to experts in topology already working in GMS-theory, as well as to those who want to enrich their research in other topics by linking it to GMS-theory. The book will also help to develop new original special courses in general topology.

Alexander V. Arhangel'skiĭ
Distinguished Professor Emeritus, Ohio University,
and Professor of Moscow University
April 13, 2016

Preface

The uniqueness of this book lies in describing generalized metric spaces by means of mappings. Back in 1961, Alexandroff (another translation of Aleksandrov) [3] proposed the idea of using the mapping method to study spaces in the first Prague Topological Symposium. The survey paper "Mappings and spaces" written by Arhangel'skiĭ [31] in 1966 inherited and developed the idea. We were greatly interested in this paper. Professors L. Wu and B. Chen then translated it into Chinese (originally in Russian) and published it in "Mathematics" (1981-1982), and wanted to arouse the interest of Chinese scholars. For a comprehensive introduction to the theory of generalized metric spaces, we recommend the books written by Burke, Lutzer [88] and Gruenhage [162], and two chapters written by Nagata [378] and Tamano [449] in the book "Topics in General Topology" [364].

There are roughly three perspectives of investigating spaces by using mappings:

(1) Which classes of generalized metric spaces can be represented as images or preimages of metric spaces under certain mappings? For example, the M-spaces introduced by Morita [360] for investigating the normality of product spaces can be expressed as preimages of metric spaces under quasi-perfect mappings. This has opened up a new way of investigating M-spaces and established connection between this class of spaces and metric spaces.

(2) What are the intrinsic characterizations of images of metric spaces under certain mappings? For example, the closed images of metric spaces (usually called Lašnev spaces) are characterized, by Foged [130], as regular Fréchet-Urysohn spaces with a σ-hereditarily closure-preserving k-network. Thus, it can be compared with the Burke-Engelking-Lutzer metrization theorem [87] and one can connected these spaces with some generalized metric spaces defined by k-networks, for example $\aleph$-spaces etc.

(3) Certain generalized metric spaces are preserved under what kinds of mappings? Take the class of metrizable spaces as an example, by the Hanai-Morita-Stone theorem [363, 441], we know that metrizability is invariant under perfect mappings. Michael [336] further proved that metrizability is invariant under countably bi-quotient closed mappings, which shows the charm of bi-quotient mappings.

These simple examples in the above three aspects reveal great colorfulness and attractiveness of infiltrations of mapping method for investigating generalized metric spaces.

Following the ideas and methods of Alexandroff-Arhangel'skii, this book presents the relevant results in the theory of generalized metric spaces in the past three decades from the perspective of mappings particularly the achievements of Chinese scholars in recent years. As an interesting scientific research reading material, it further elaborates these ideas and methods.

This book is also suitable as an elective course material or a teaching reference book for the graduates and senior undergraduates of mathematics major, or as a reading material for graduate students of general topology. It also can be used as a reference book for mathematicians and scientific researchers in other fields.

Guoshi Gao
Soochow University,
September 1, 1992

Preface to the English Edition

The concept of paracompactness introduced by Dieudonné [112] in 1944 is a significant sign of general topology to enter the peak period. The extraordinary Bing-Nagata-Smirnov metrization theorem [62, 370, 428], established in the years 1950 and 1951, created a fundamental change for the full exploration of the nature of metrizability. Meanwhile, the theorem disclosed a bright future of the investigation on properties of generalized metric spaces. The deep study on paracompactness and metrizability performs the prelude to the research on the theory of generalized metric spaces.

In 1961, at the international topological symposium named "General Topology and its Relationship with Modern Analysis and Algebra" in Prague, Alexandroff [3] put forward the idea of investigating spaces by mappings, namely, to connect various classes of spaces by using mappings as a linkage. In this way, the fundamental research framework and whole structure of the theory of topological spaces can be reflected by the relationships between mappings and spaces. Mappings thus became a powerful tool for revealing the internal properties of various classes of spaces. In 1966, Arhangel'skiĭ [31] published the historical literature titled "Mappings and Spaces" which presented a series of constructive concrete steps of how to operate the Alexandroff idea. As a milepost in the road of the vigorous development of general topology, it ushered in an innovation era for investigating spaces by mappings. Since then, the Alexandroff idea became an indispensable method for the research. It promoted a rapid development of general topology, particularly the theory of generalized metric spaces. In a word, the Alexandroff idea of mutual classifications of spaces and mappings has constituted an important part of contemporary general topology [4].

According to Alexandroff and Arhangel'skiĭ, the core of investigating spaces by means of mappings is to establish the extensive connections, with the aid of mappings, between the class of metric spaces and classes of spaces having specific topological properties, to study the intrinsic characterizations of images of metric spaces under various mappings, and to discuss which kinds of mappings can preserve certain classes of spaces. This framework determines the goals of this book: to comprehensively describe characterizations of images of metric spaces under various mappings and to establish mapping theorems for several important generalized metric spaces.

The most prominent feature of this book is to discuss the theory of generalized metric spaces in a systematic way from the perspective of mappings. Through the mapping theorems of generalized metric spaces, this book tries to point out that the principle of classifying spaces by mappings is undeniable a powerful research tool that has decisive significance in general topology, by which one can peep the full linkage between spaces and mappings. Most of the contents of this book are from hundreds of research papers on spaces and mappings published in recent 50 years. While paying attention to the innovation and unique features of the contents, this book highlights the outstanding achievements of Chinese scholars in recent years. The authors hope this book can guide readers to grasp the principle of classifying spaces by mappings and catch up the new developments of the theory of generalized metric spaces. We also hope this book can provides solid support for readers in their further research work and can magnify the impacts from China's general topological circles to the world.

This book is composed of three chapters, two appendices and a list of more than 400 references. To provide necessary preparation for the descriptions of relationships between spaces and mappings in the following two chapters, Chapter 1 briefly introduces some basic concepts of generalized metric spaces, the properties of fundamental operations on these spaces and their simple characterizations. By means of several families of sets with specific properties and the concepts of bases or their generalizations, Chapter 2 presents the intrinsic characterizations of images or preimages of metric spaces under quotient mappings, pseudo-open mappings, countably bi-quotient mappings, open mappings, closed mappings and compact-covering mappings or under these mappings with some additional conditions, for example the fibers are separable or compact. Chapter 3 introduces several classes of generalized metric spaces defined by specific bases, weak bases, k-networks, networks and (mod k)-networks etc., such as M-spaces, p-spaces, g-metrizable spaces, $\aleph$-spaces, k-semi-stratifiable spaces, σ-spaces, Σ-spaces and so on. Characterizations and mapping theorems of these spaces are also obtained in this chapter. There are two appendices at the end of this book. The first one aims to help readers to better understand some results on the covering properties used in the text. The second one aims to help readers to obtain a more comprehensive vision of the theory of generalized metric spaces, particularly to help them clearly recognize the place of Alexandroff's idea in contemporary general topology.

The first two editions of this book were written in Chinese and published in 1995 and 2007 respectively [271, 282]. They have played an active role on the development of the theory of spaces and mappings. The primary goal of this new edition is to update recent developments in the area and to continue to revise the text for the sake of clarity and accessibility for readers. In order to arouse interest about the theory of generalized metric spaces and promote contributions of Chinese mathematicians in a broader international society, this edition is published in English.

We benefited from the input of many talented people in writing this book. The most important person in our academic career is Professor G. Gao①. He is our mentor. His outstanding work in spaces and mappings and his actively promoting Alexandroff-Arhangelskiĭ's ideas have created opportunities for us to deeply engage in the study of this subject [140, 142]. We are very grateful to Prof. Gao's thorough coaching, invaluable advice, altruistic helps and constant encouragements since 1979. He has laid down the learning and research foundations for us, pointed out the direction of further exploration to us and given us courage and confidence in our research. This book reflects Professor Gao's research style and academic thinking to a certain extent. We also thank Professor Alexander V. Arhangel'skiĭ for his valuable comments on improving the writing of this book.

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Shou Lin   Ziqiu Yun
Minnan Normal University,
Soochow University,
May 12, 2016

①G. Gao, see "Biographies of Modern Chinese Mathematicians in the 20th Century, V. 3", edited by M. Chen, Jiangsu Education Press, Nanjing, 1998, 287–297.

Contents

Chapter 1

The origin of generalized metric spaces

Metrization theory is the core in the study of general topology and the theory of generalized metric spaces is a generalization of this theory. By the opinion of Burke and Lutzer [88], the theory of generalized metric spaces has its roots mainly in three problems: the metrization problem, the problem of paracompactness in product spaces and the mutual classification problem of spaces and mappings suggested by Alexandroff in 1961 [3].

What is a "*generalized metric space*"? Perhaps, all the topological properties which are weaker than those for metrization can be called generalized metric properties. However, this interpretation is too broad. Roughly speaking, the class of generalized metric spaces are such classes of spaces that they posses some properties which are useful in characterizing metrizability, and inherit many delicate natures of metric spaces, and some of the theory or techniques of metric spaces can be carried over to these classes [162, 193].

Hodel [193] pointed out: "There are a number of reasons why generalized metrizable spaces are worthy of study. Perhaps the most important reason is that such classes increase our understanding of the metrizable spaces. But in addition, topologists are continually seeking broader classes of spaces for which especially important results hold." For example, the Katětov-Morita dimension theorem, the Dugundji extension theorem and the Borsuk homotopy extension theorem etc. can be established in these spaces. Because of these reasons, the theory of generalized metric spaces has been one active research direction in general topology since the 1960's. A large number of problems of this theory which are formed by the mutual blending of other branches, such as axiomatic set theory, mathematical logic, combinatorics mathematics, functional analysis, topological algebra, dynamic system and computer science etc., have been included in the books "Open Problems in Topology" [342], "Open Problems in Topology II" [392] and "Problems from Topology Proceedings" [391]. The achievements of the theory of generalized metric spaces between 1960 and 2015 have been summarized in some important works, such as Arhangel'skiĭ [31, 34], Burke and Lutzer [88], G. Gao [140, 141], Gruenhage [162–164, 166], Hodel [193], Kodama and Nagami [230], S. Lin [282, 285], Morita and Nagata [364], Nagata [373,376]. Numbers of challenging problems proposed by many scholars, together with some of long-term unresolved problems, have become the birthplaces of further development of the theory of generalized metric spaces. Just as Hodel [193] pointed out after summing up the staggering achievement of the theory of generalized metric spaces: "But more important perhaps is the fact that the study of generalized metrizable spaces is by no means complete; rather, it continues to grow with many new and important results appearing every year."

In this chapter, we derive most of the generalized metric spaces that will be discussed in this book from three perspectives: distance functions, bases or their extensions and generalized countable compactness. At the same time, we give some simple characterizations of these spaces, describe their basic operation properties, such as hereditary properties, productive properties and properties preserved under topological sums, and discuss their preliminary relationships with some other classes of spaces.

1.1 Notations and terminologies

Through the book, spaces are all Hausdorff topological spaces.

In this section, we first give definitions of the notations and terminologies used frequently in this book, and then list several classic results.

1.1.1 Sets of real numbers and cardinal numbers

We denote the real line by $\mathbb{R}$. The sets of natural numbers, positive integers, rational numbers, irrational numbers and nonnegative real numbers are denoted as ω, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{P}$ and $\mathbb{R}^+$ respectively. The letter ω also denotes the smallest infinite ordinal. The unit closed interval is denoted by $\mathbb{I}$. Denote $\mathbb{S}_1 = \{0\} \cup \{1/n : n \in \mathbb{N}\}$. The cardinalities of $\mathbb{N}$ and $\mathbb{R}$ are denoted by $\aleph_0$ and $\mathfrak{c}$ respectively. The smallest uncountable cardinality (resp. ordinal) is denoted as $\aleph_1$ (resp. ω_1).

1.1.2 Operations on subsets of topological spaces

The topology and the set of closed subsets of a space X are denoted by $\tau(X)$ and $\tau^c(X)$ respectively, and when there is no ambiguity, $\tau(X)$ and $\tau^c(X)$ are denoted as τ and τ^c respectively. Let A and (Y, τ') be a subset and a subspace of X respectively and let $Z \subset Y$. Then

the closure of A in X is denoted as $\overline{A}$ or $\mathrm{cl}(A)$;
the interior of A in X is denoted as A° or $\mathrm{int}(A)$;
the boundary of A in X is denoted as ∂A;
the set of accumulation points of A in X is denoted as A^d;
the closure of Z in Y is denoted as $\mathrm{cl}_Y(Z)$ or $\mathrm{cl}_{\tau'}(Z)$;

the interior of Z in Y is denoted as $\mathrm{int}_Y(Z)$ or $\mathrm{int}_{\tau'}(Z)$.

1.1.3 Families of sets in topological spaces

In the sequel, a family always means a family of sets. For any space X, denote

$\mathscr{K}(X) = \{K \subset X : K$ is a compact set in $X\}$, and

$\mathscr{S}(X) = \{S \subset X : S$ is a convergent sequence in X with its limit point$\}$.

Each finite subset of X is regarded as a trivial convergent sequence. A sequence $\{x_n\}$ in X is called a *nontrivial sequence*, if $x_n \neq x_m$ whenever $n \neq m$.

Let $\mathscr{P}$ be a family in X. Then

$\mathscr{P}^{<\omega} = \{\mathscr{F} \subset \mathscr{P} : \mathscr{F}$ is finite$\}$;

$\mathscr{P}^F = \{\cup\mathscr{F} : \mathscr{F} \in \mathscr{P}^{<\omega}\}$;

$\cup\,\mathscr{P} = \cup\{P : P \in \mathscr{P}\}$ is called the union of $\mathscr{P}$;

$\cap\,\mathscr{P} = \cap\{P : P \in \mathscr{P}\}$ is called the intersection of $\mathscr{P}$;

$\mathscr{P}^- = \overline{\mathscr{P}} = \{\overline{P} : P \in \mathscr{P}\}$ is called the closure of $\mathscr{P}$;

$\mathscr{P}^\circ = \{P^\circ : P \in \mathscr{P}\}$ is called the interior of $\mathscr{P}$;

$\bigoplus\mathscr{P} = \bigoplus\{P : P \in \mathscr{P}\}$ is called the topological sum of $\mathscr{P}$.

For every $A \subset X$ and $x \in X$, denote

$$(\mathscr{P})_A = \{P \in \mathscr{P} : P \cap A \neq \varnothing\}, \quad (\mathscr{P})_x = (\mathscr{P})_{\{x\}};$$
$$\mathrm{st}(A, \mathscr{P}) = \cup(\mathscr{P})_A, \quad \mathrm{st}(x, \mathscr{P}) = \cup(\mathscr{P})_x;$$
$$\mathrm{st}^{n+1}(A, \mathscr{P}) = \mathrm{st}(\mathrm{st}^n(A, \mathscr{P}), \mathscr{P}),\ n \in \mathbb{N};$$
$$\mathscr{P}_{|A} = \{P \cap A : P \in \mathscr{P}\}.$$

If $\mathscr{F}$ is also a family in X, denote $\mathscr{P} \wedge \mathscr{F} = \{P \cap F : P \in \mathscr{P}, F \in \mathscr{F}\}$. The definition of $\bigwedge_{\alpha\in\Gamma}\mathscr{P}_\alpha$ is similar.

1.1.4 Mappings on spaces

Assume that X and Y are spaces and $f : X \to Y$ is a mapping.

Suppose $A \subset X$, the restriction of f to A is the function $f_{|A} : A \to f(A)$ whose rule is $f_{|A}(x) = f(x)$, for each $x \in A$.

For each $B \subset Y$, $f_{|f^{-1}(B)}$ is denoted by f_B and called the restriction of f to B.

If $\mathscr{P}$ is a family in X, then $f(\mathscr{P}) = \{f(P) : P \in \mathscr{P}\}$ is called the image of $\mathscr{P}$ under f.