THEORETICAL ELASTICITY

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PREFACE

THIS book is mainly concerned with three aspects of elasticity theory which have attracted considerable attention in recent years, and is not intended to be an exhaustive treatise. Many important topics, such as the torsion and flexure of beams, energy methods, and the theory of elastic stability, are omitted because they have already been extensively discussed in other books. The three main topics considered here are finite elastic deformations, complex variable methods for two-dimensional problems for both isotropic and aeolotropic bodies, and shell theory, the latter topics being confined to classical infinitesimal elasticity. In addition, some mention is made of three-dimensional problems for isotropic and hexagonally aeolotropic bodies. Throughout the book emphasis is placed on the use of general tensor notations in which general theories can be expressed in an elegant and compact form, and which are of considerable help in the solution of special problems, particularly for finite deformations. Vector notations are also used with tensors whenever appropriate.

A number of books on tensor analysis are available, but since workers in the field of elasticity are still often unfamiliar with these notations we have included a summary of tensors in Chapter I. By restricting the discussion to a three-dimensional Euclidean space it is possible to present a comparatively simple account of the relevant theory. The main properties of two-dimensional surfaces are then deduced by regarding such surfaces as being embedded in three-dimensional Euclidean space. Tensor analysis can then be extended to general Riemannian spaces without difficulty, but this extension is not required here. The advantages of the presentation of tensor analysis from a restricted point of view seems to be sufficient to justify the sacrifice of some generality.

Chapter II contains an account of the general theory of elasticity for finite deformations, using the notations of Chapter I. Special attention is given to the formulation of stress-strain relations for an isotropic body. Chapter III contains solutions of a number of special problems, mostly for incompressible isotropic bodies, the majority being obtained in a general form which is independent of the choice of strain-energy function. A number of these solutions have been found to have practical application for rubber-like materials, but discussion of such applications is not included in the book.

In Chapter IV a theory of small deformations, which are superposed on finite deformations, is given, again making no assumptions about the form of the strain-energy function. This theory is available for both compressible and incompressible bodies and a number of special problems are solved. The advantages of tensor notations are again evident in this chapter.

Chapter V contains the classical infinitesimal theory of elasticity which is deduced as a special case of the general theory developed in Chapter II. Once again, tensor notations are used, so that specialization to particular coordinate systems is then a straightforward matter. In this chapter stress-strain relations are deduced for aeolotropic as well as isotropic bodies. The chapter closes with the solution of some three-dimensional problems for both isotropic and hexagonally aeolotropic bodies.

Chapters VI and VII deal with plane strain and plate theories for both isotropy and aeolotropy and are first developed in tensor notation. By specialization of this general form of the two-dimensional theories it is possible to introduce complex variable notations in a consistent and natural manner, so that complex combinations of stresses appear by using tensor transformations from rectangular-cartesian to complex coordinates. Two-dimensional problems are discussed in Chapter VIII for isotropic bodies, and in Chapter IX for aeolotropic bodies for plane strain, and for plates deformed by forces in their planes, using some powerful and elegant tools of complex function theory. Problems of transverse flexure of plane plates are not discussed in detail since many of these problems are analytically similar to those occurring in plane strain. In Chapter VII, however, an extension of the classical theory of flexure of isotropic plates, due to Reissner, is considered, and a special problem is solved in Chapter VIII, using this theory.

Some of the general methods of solution of two-dimensional problems given in Chapter VIII are due to Muskhelishvili and other Russian writers, and this chapter (and § 1.14 to § 1.21) owes much to a book by Muskhelishvili, *Singular Integral Equations* (Moscow, 1946), which was translated by J. R. M. Radok and W. G. Woolnough (Australia, 1949). At the time of writing, this book, translated by J. R. M. Radok, has received wider publication by P. Noordhoff (Groningen, Holland, 1953). In addition, another important book by Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, translated by J. R. M. Radok, has just been published by P. Noordhoff. Reference is made to the Russian version of this book in footnotes to Chapter VI but the book has not been available to the present writers.

In footnotes on page 185 of Chapter VI references are given to pioneer work by Lechnitzky in two-dimensional problems for aeolotropic bodies, and references to work by other Russian writers are contained in the paper by Sokolnikoff which is quoted on page 185. The work on aeolotropic materials by Russian writers has not been available to the present authors so that it has not been possible to refer to it adequately in this book. For this reason reference is often made to papers which have appeared in British journals but it is recognized that Russian authors may frequently claim priority.

The last chapters of the book, X–XIV, are devoted to the theory of shells; here emphasis is placed on the formulation of a general theory and only a few special problems are discussed. Once more the value of tensor notations is evident. The theory of shells given in Chapter X differs in some respects from existing theories. Attention is restricted to a first approximation, which, it is believed, is satisfactory for many problems which arise in practice.

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A. E. G.
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CONTENTS

I

MATHEMATICAL PRELIMINARIES

THIS chapter contains a summary of some important definitions, relations and formulae of vector and tensor calculus, functions of a complex variable, and Fourier integrals, which are essential for our treatment of the theory of elasticity. We have not included proofs of all theorems and formulae but it is hoped that sufficient details are given so that the reader may understand the remaining chapters without being forced to make constant reference to other mathematical books.†

1.1. Indicial notation. Summation convention. Kronecker delta

Consider symbols which are characterized by one or several indices which may be either subscripts or superscripts,‡ such as A_i, B^i, A_{ij}, B^i_j, etc. Sometimes it is necessary to decide the order of the indices when subscripts and superscripts occur together and then, for example, we write $A^i_{\cdot j}$ where the dot before j shows that j is the second index and i the first.

Unless otherwise stated Latin indices represent the numbers 1, 2, 3. Thus, A^i represents any one of the three elements A^1, A^2, A^3 and A_{ij} represents any one of the nine elements A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}, A_{31}, A_{32}, A_{33}.

Systems of elements which, like A^i, depend on one index only, are called systems of the first order, and the separate terms A^1, A^2, A^3 are called the components of the system. Systems of the first order have one or other of the two forms

$$A^i,\ B_i.$$

† More detailed treatments of vector and tensor calculus are given, e.g., by A. Duschek and A. Hochrainer, *Grundzüge der Tensorrechnung in Analytischer Darstellung*, vol. i (1948), vol. ii (1950), Vienna; L. A. Eisenhart, *An Introduction to Differential Geometry* (Princeton, 1947); A. J. McConnel, *Applications of the Absolute Differential Calculus* (London and Glasgow, 1947); F. D. Murnaghan, *Introduction to Applied Mathematics* (New York, 1948); J. L. Synge and A. Schild, *Tensor Calculus* (Toronto, 1949); I. S. Sokolnikoff, *Tensor Analysis* (New York, 1951); C. E. Weatherburn, *An Introduction to Riemannian Geometry and the Tensor Calculus* (Cambridge, 1938). The method of presentation of tensor calculus given in this chapter differs, however, in some respects from that of other writers. §§ 1.14 to 1.21 on complex variable theory were written with the hel of a book by N. I. Muskhelishvili, *Singular Integral Equations* (Moscow, 1946), translated by J. R. M. Radok and W. G. Woolnough (Australia, 1949).

Reference may be made to E. C. Titchmarsh, *Theory of Fourier Integrals* (Oxford, 1937), for the results of § 1.22.

‡ It is understood that indices as superscripts are not taken as powers.

Systems of the second order depend on two indices and can be of the three types

$$A_{ij},\ A^{i}_{\cdot j} \text{ or } A_j{}^{\cdot i},\ A^{ij},$$

and there are nine components in each system. Similarly, we have systems of the third and higher orders.

A single element ϕ, which has no indices, is called a system of order zero.

Expressions which consist of a sum are formed by the following summation convention, unless stated otherwise. Any term in which the same index (subscript or superscript) appears twice stands for the sum of all such terms obtained by giving this index its complete range of values. The following examples illustrate this convention:

$$\left.\begin{aligned} A^iB_i &= \sum_{i=1}^{3} A^iB_i = A^1B_1+A^2B_2+A^3B_3 \\ A^{i}_{\cdot i} &= \sum_{i=1}^{3} A^{i}_{\cdot i} = A^{1}_{\cdot 1}+A^{2}_{\cdot 2}+A^{3}_{\cdot 3} \end{aligned}\right\}, \qquad (1.1.1.)$$

$$\left.\begin{aligned} A_{ij}x^ix^j = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}x^ix^j &= A_{11}x^1x^1+A_{12}x^1x^2+A_{13}x^1x^3+ \\ &\quad +A_{21}x^2x^1+A_{22}x^2x^2+A_{23}x^2x^3+ \\ &\quad +A_{31}x^3x^1+A_{32}x^3x^2+A_{33}x^3x^3 \end{aligned}\right\}. \qquad (1.1.2)$$

Since the repeated index is summed it follows that we may substitute for the particular letter used any other letter without altering the value of the expansion. Thus

$$A^iB_i = A^jB_j.$$

No summation is carried out if the same index is repeated more than twice as, for example, in $A^{ij}B_{ii}$. On the other hand, when more than one summation is necessary, different summation indices are taken, as in example (1.1.2).

It may be mentioned that it does not, of course, follow from an equation of the type

$$A^iB_i = A^iC_i$$

that $B_i = C_i$, since both sides of the equation represent the sums of three different terms.

A special meaning is given to the symbols δ^i_j which are called Kronecker deltas. The Kronecker deltas have the following values:

$$\left.\begin{aligned} \delta^i_j &= 0 \quad (i \neq j) \\ \delta^i_j &= 1 \quad (i = j,\ j \text{ not summed}) \end{aligned}\right\}. \qquad (1.1.3)$$

We therefore have

$$\delta^1_2 = \delta^1_3 = \delta^2_1 = \delta^2_3 = \delta^3_1 = \delta^3_2 = 0,$$

$$\delta^1_1 = \delta^2_2 = \delta^3_3 = 1.$$

The Kronecker delta is sometimes called the substitution operator since, for example,

$$\delta^i_j a^j = a^i, \qquad \delta^i_k a_{ij} = a_{kj}. \tag{1.1.4}$$

1.2. Transformations of coordinates

We denote by θ^i three independent variables whose differentials are $d\theta^i$. We also introduce the convention that partial derivatives of a function with respect to the independent variables are denoted by a comma. For example,

$$A_{,i} = \frac{\partial A}{\partial \theta^i},$$

$$A_{jk,i} = \frac{\partial A_{jk}}{\partial \theta^i},$$

$$A^{jk}{}_{,i} = \frac{\partial A^{jk}}{\partial \theta^i}.$$

Let us now suppose that the variables θ^i are transformed into a set of new variables $\bar{\theta}^i$ by any arbitrary single-valued functions of the form

$$\bar{\theta}^i = \bar{\theta}^i(\theta^1, \theta^2, \theta^3). \tag{1.2.1}$$

We assume that the arbitrary functions possess derivatives up to any order required and also that the transformation (1.2.1) is reversible. We therefore have the inverse transformation

$$\theta^i = \theta^i(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3), \tag{1.2.2}$$

and we assume that the functions θ^i are also single-valued.

The transformation of differentials $d\bar{\theta}^i$ and $d\theta^i$ follows immediately from (1.2.1) and (1.2.2), so that

$$\left.\begin{aligned} d\bar{\theta}^i &= \bar{c}^i_j d\theta^j \\ d\theta^i &= c^i_j d\bar{\theta}^j \end{aligned}\right\}, \tag{1.2.3}$$

where

$$\bar{c}^i_j = \frac{\partial \bar{\theta}^i}{\partial \theta^j}, \qquad c^i_j = \frac{\partial \theta^i}{\partial \bar{\theta}^j}. \tag{1.2.4}$$

The functions in (1.2.4) are related by the equations

$$\bar{c}^i_k c^k_j = c^i_k \bar{c}^k_j = \delta^i_j, \tag{1.2.5}$$

from which the values of $\bar{c}^i_j$ can be calculated when c^i_j are known, and vice versa, provided that the functional determinant

$$\bar{c} = |\bar{c}^i_j| \neq 0.$$

This last condition holds because our transformations are assumed to be reversible.

From (1.2.3) we see that the transformation of the differentials is a linear one, while the transformation of the variables θ^i in (1.2.1) is, of course, not linear in general.

1.3. Invariants. Tensors

Consider a system T of functions (of any order) whose components are defined in the general set of variables θ^i and are functions of $\theta^1, \theta^2, \theta^3$. If the variables θ^i can be changed to $\bar{\theta}^i$ by equations (1.2.1) we can define new components of T in the general variables $\bar{\theta}^i$ which are functions of $\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3$, and if the components of T in the two sets of variables are related by certain rules, which we now examine, the system of functions T is called a tensor.

I. A system of order zero may be defined to have a single component ϕ in the variables θ^i, and a single component $\bar{\phi}$ in the variables $\bar{\theta}^i$. If

$$\bar{\phi}(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3) \equiv \phi(\theta^1, \theta^2, \theta^3), \tag{1.3.1}$$

that is, numerically equal at corresponding values of $\bar{\theta}^i$, θ^i, then the functions ϕ and $\bar{\phi}$ of the θ^i's and $\bar{\theta}^i$'s respectively are the components in their respective variables of a *tensor of order zero*. This system is also called a *scalar invariant* or *scalar*. For brevity we shall say that ϕ (or $\bar{\phi}$) is a scalar.

II. A system of order one may be defined to have three components A^i in the variables θ^i and three components $\bar{A}^i$ in the variables $\bar{\theta}^i$. If

$$\bar{A}^i = \bar{c}^i_j A^j, \tag{1.3.2}$$

then the functions $A^i(\theta^1, \theta^2, \theta^3)$ and $\bar{A}^i(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3)$ are the components in their respective variables of a *contravariant tensor of order one*. For brevity we shall say that A^i (or $\bar{A}^i$) is a contravariant tensor of order one.

III. A system of order one may be defined to have three components A_i in the variables θ^i and three components $\bar{A}_i$ in the variables $\bar{\theta}^i$. If

$$\bar{A}_i = c^j_i A_j, \tag{1.3.3}$$

then the functions $A_i(\theta^1, \theta^2, \theta^3)$ and $\bar{A}_i(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3)$ are the components in their respective variables of a *covariant tensor of order one*. For brevity we shall say that A_i (or $\bar{A}_i$) is a covariant tensor of order one.

IV. A system of order two may be defined to have nine components A^{ij} in the variables θ^i and nine components $\bar{A}^{ij}$ in the variables $\bar{\theta}^i$. If

$$\bar{A}^{ij} = \bar{c}^i_m \bar{c}^j_n A^{mn}, \tag{1.3.4}$$

then the functions $A^{ij}(\theta^1, \theta^2, \theta^3)$ and $\bar{A}^{ij}(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3)$ are the components in their respective variables of a *contravariant tensor of order two.* For brevity we shall say that A^{ij} (or $\bar{A}^{ij}$) is a contravariant tensor of order two.

V. A system of order two may be defined to have nine components A_{ij} in the variables θ^i and nine components $\bar{A}_{ij}$ in the variables $\bar{\theta}^i$. If

$$\bar{A}_{ij} = c^m_i c^n_j A_{mn}, \tag{1.3.5}$$

then the functions $A_{ij}(\theta^1, \theta^2, \theta^3)$ and $\bar{A}_{ij}(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3)$ are the components in their respective variables of a *covariant tensor of order two.* For brevity we call A_{ij} (or $\bar{A}_{ij}$) a covariant tensor of order two.

VI. A system of order two may be defined to have nine components $A^{i}_{\cdot j}$ (or $A_j{}^{\cdot i}$) in the variables θ^i and nine components $\bar{A}^{i}_{\cdot j}$ (or $\bar{A}_j{}^{\cdot i}$) in the variables $\bar{\theta}$. If

$$\left.\begin{aligned} \bar{A}^{i}_{\cdot j} &= \bar{c}^i_m c^n_j A^{m}_{\cdot n} \\ \bar{A}_j{}^{\cdot i} &= \bar{c}^i_m c^n_j A_n{}^{\cdot m} \end{aligned}\right\}, \tag{1.3.6}$$

then the funtions $A^{i}_{\cdot j}(\theta^1, \theta^2, \theta^3)$ {or $A_j{}^{\cdot i}(\theta^1, \theta^2, \theta^3)$} and $\bar{A}^{i}_{\cdot j}(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3)$ {or $\bar{A}_j{}^{\cdot i}(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3)$} are the components in their respective variables of a *mixed tensor of order two.* For brevity we call $A^{i}_{\cdot j}$ (or $\bar{A}^{i}_{\cdot j}$), and $A_j{}^{\cdot i}$ (or $\bar{A}_j{}^{\cdot i}$), mixed tensors of order two.

The order of the tensor is denoted by the number of indices and the type of tensor (contravariant, covariant, or mixed) by the position of the indices, so that we frequently refer to tensors (e.g. A^i, A_{ij}, $A_j{}^{\cdot i}$), and omit explicit reference to the order or type.

In a similar way tensors of higher orders may be formed. For example, mixed components $A^{i}_{\cdot jk}(\theta^1, \theta^2, \theta^3)$ and $\bar{A}^{i}_{\cdot jk}(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3)$, in their respective variables, of a tensor of order three, are related by the law

$$\bar{A}^{i}_{\cdot jk} = \bar{c}^i_r c^s_j c^t_k A^{r}_{\cdot st}. \tag{1.3.7}$$

Again, we often omit the word component and refer to a mixed tensor $A^{i}_{\cdot jk}$ (or $\bar{A}^{i}_{\cdot jk}$) of order three, or simply, a tensor $A^{i}_{\cdot jk}$, and similarly for tensors of any order or type.

From (1.2.3) we see that the differentials $d\theta^i$ transform according to the law for contravariant tensors so that the position of the upper index is justified. The variables θ^i themselves are in general neither contravariant nor covariant and the position of their index must be recognized as an exception. In future the index in non-tensors will be placed either above or below according to convenience. For example, we shall use either θ^i or θ_i.

We notice that each component of a tensor in the new variables is a linear combination of the components in the old variables. Consequently, if all components of a tensor are zero in one system of variables they are also zero in all other systems of variables which can be obtained by transformations of the type (1.2.1) and (1.2.2).

It also follows immediately from the definitions (1.3.1) to (1.3.7) and the relations (1.2.4) and (1.2.5) that all functions of the form

$$A^i B_i, \quad A^{ij} B_{ij},$$

etc., are invariants, if, for example, using an obvious notation, we define

$$\overline{A^i B_i} = \bar{A}^i \bar{B}_i, \qquad \overline{A^{ij} B_{ij}} = \bar{A}^{ij} \bar{B}_{ij}.$$

We observe that components of tensors satisfy the group property under general transformations of variables. In other words, if the transformation relations exist when the variables are changed from θ_i to $\bar{\theta}_i$, and when they are changed from θ_i to θ'_i, then they also exist when the variables are changed from θ'_i to $\bar{\theta}_i$. This property follows easily from the forms of the transformations.

We may notice at this point the *arbitrary* character of tensors. We may take as the components of a tensor in a given set of variables any set of functions of the requisite number. We then define components in any *general* system of variables by the equations expressing the law of transformation for that particular tensor. Because of the group property we then know that the components of our tensor expressed in a general system of variables will always transform according to our tensor rules.

1.4. Addition, multiplication, and contraction of tensors

The operations of addition and subtraction of tensors apply only to tensors of the same order and type and lead to tensors of the same order and type. For example, if A^i and B^i are contravariant tensors of order one then C^i defined by the equation

$$C^i = A^i + B^i$$

is also a contravariant tensor of order one. Or again, the difference of $A^i_{.jk}$ and $B^i_{.jk}$ leads to a mixed tensor $C^i_{.jk}$ of the third order where

$$C^i_{.jk} = A^i_{.jk} - B^i_{.jk}.$$

The equations $$A^{ij} = B^{ij}$$

are said to form a *tensor* equation, that is, if they are true in one system of variables they are true in all other systems. This follows at once from the definition of a tensor since components in one set of variables define one unique tensor.

The *multiplication* of tensors leads to tensors of higher orders. For example, if we multiply A_i and B_{jk} we get the tensor

$$C_{ijk} = A_i B_{jk}$$

of order three.

Another operation applied to tensors is the operation of *contraction*. Let us consider the mixed tensor $A^{ij}_{\cdot\cdot k}$ of order three. If we make the indices k and i the same so that the tensor becomes $A^{ij}_{\cdot\cdot i}$, and if we remember that the repeated index is to be summed, we see that the new system has its order reduced by two. It can be shown from (1.3.2) to (1.3.7) that the new system forms a contravariant tensor of order one.

We see that any combination of the operations of addition, subtraction, multiplication, and contraction on tensors produces new tensors, and these operations are called tensor operations. We often recognize the tensorial character of a system of functions by observing that they are formed by a combination of these operations on known tensors.

Finally, we mention another important rule concerning tensors. If, for example, we have a system of nine functions $A_{(ij)}$ such that

$$A_{(ij)} a^i = B_j$$

for *every* contravariant tensor a^i, where B_j is known to be a covariant tensor, then

$$A_{(ij)} = A_{ij}$$

is a tensor. This result can be generalized to apply to tensors of any order and type.

1.5. Symmetric and skew-symmetric tensors

If we are given the tensors A^{ij}, A_{ij} it may happen that each component is unaltered in value when the indices are interchanged, so that

$$A^{ij} = A^{ji}, \qquad A_{ij} = A_{ji}. \tag{1.5.1}$$

The tensors are then said to be *symmetric*. More generally, a tensor of any order is said to be symmetric in two subscripts or superscripts if it is unaltered when the two indices are interchanged, and the tensor is completely symmetric if the interchange of any subscripts or superscripts leaves it unaltered. The tensor A_{ijk} of the third order which is completely symmetric will, for example, satisfy the relations

$$A_{ijk} = A_{ikj} = A_{jik} = A_{jki} = A_{kij} = A_{kji}. \tag{1.5.2}$$