PRINCIPLES OF REAL ANALYSIS

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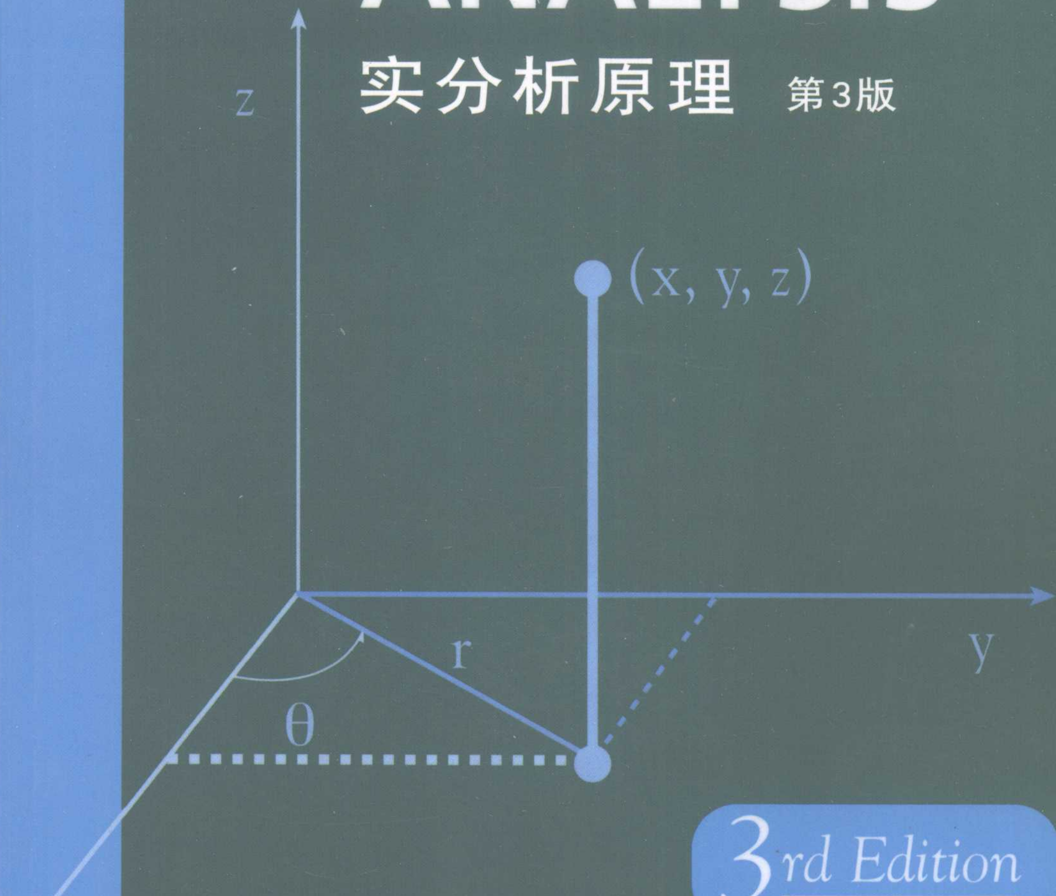

3rd Edition

Charalambos D. Aliprantis
Owen Burkinshaw

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PRINCIPLES OF REAL ANALYSIS
Third Edition

CHARALAMBOS D. ALIPRANTIS
Departments of Economics and Mathematics
Purdue University

and

OWEN BURKINSHAW
Department of Mathematical Sciences
Indiana University, Purdue University, Indianapolis

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PRINCIPLES OF
REAL ANALYSIS
Third Edition

To our wives
Bernadette and Betty,
and to our children
Claire, Dionissi, and Mary.

PREFACE

This is the third edition of *Principles of Real Analysis*, first published in 1981. The aim of this edition is to accommodate the current needs for the traditional real analysis course that is usually taken by the senior undergraduate or by the first year graduate student in mathematics. This edition differs substantially from the second edition. Each chapter has been greatly improved by incorporating new material and by rearranging the old material. Moreover, a new chapter (Chapter 6) on Hilbert spaces and Fourier analysis has been added.

The subject matter of the book focuses on measure theory and the Lebesgue integral as well as their applications to several functional analytic directions. As in the previous editions, the presentation of measure theory is built upon the notion of a semiring in connection with the classical Carathéodory extension procedure. We believe that this natural approach can be easily understood by the student. An extra bonus of the presentation of measure theory via the semiring approach is the fact that the product of semirings is always a semiring while the product of σ-algebras is a semiring but not a σ-algebra. This simple but important fact demonstrates that the semiring approach is the natural setting for product measures and iterated integrals.

The theory of integration is also studied in connection with partially ordered vector spaces and, in particular, in connection with the theory of vector lattices. The theory of vector lattices provides the natural framework for formalizing and interpreting the basic properties of measures and integrals (such as the Radon–Nikodym theorem, the Lebesgue and Jordan decompositions of a measure, and the Riesz representation theorem). The bibliography at the end of the book includes several books that the reader can consult for further reading and for different approaches to the presentation of measure theory and integration.

In order to supplement the learning effort, we have added many problems (more than 150 for a total of 609) of varying degrees of difficulty. Students who solve a good percentage of these problems will certainly master the material of this book. To indicate to the reader that the development of real analysis was a collective effort by many great scientists from several countries and continents through the ages, we have included brief biographies of all contributors to the subject mentioned in this book.

We take this opportunity to thank colleagues and students from all over the world who sent us numerous comments and corrections to the first two editions. Special thanks are due to our scientific collaborator, Professor Yuri Abramovich, for his comments and constructive criticism during his reading of the manuscript of this edition. The help provided by Professors Achille Basile and Vinchenco Aversa of Università Federico II, Napoli, Italy, during the collection of the biographical data is greatly appreciated. Finally, we thank our students (Anastassia Baxevani, Vladimir Fokin, Hank Hernandez, Igor Kuznetsov, Stavros Muronidis, Mohammad Rahman, and Martin Schlam) of the 1997–98 IUPUI graduate real analysis class who read parts of the manuscript and made many corrections and improvements.

C. D. ALIPRANTIS and O. BURKINSHAW
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Charalambos D. Aliprantis, Owen Burkinshaw

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CHAPTER 1

FUNDAMENTALS OF REAL ANALYSIS

If you are reading this book for the purpose of learning the theory of integration, it is expected that you have a good background in the basic concepts of real analysis. The student who has come this far is assumed to be familiar with set theoretic terminology and the basic properties of real numbers, and to have a good understanding of the properties of continuous functions.

The first section of this chapter covers the fundamentals of set theory. We have kept it to the "minimum amount" of set theory one needs for any modern course in mathematics. The following two sections deal with the real and extended real numbers. Since the basic properties of the real numbers are assumed to be known, the fundamental convergence theorems needed for this book are emphasized. Similarly, the discussion on the extended real numbers is focused on the needed results. The last two sections present a comprehensive treatment of metric spaces.

1. ELEMENTARY SET THEORY

Throughout this book the following commonly used mathematical symbols will be employed:

$\forall$ means "for all" (or "for each");
$\exists$ means "there exists" (or "there is");
$\Longrightarrow$ means "implies that" (or simply "implies");
$\Longleftrightarrow$ means "if and only if."

The basic notions of set theory will be briefly discussed in the first section of this chapter. It is expected that the reader is familiar in one way or another with these concepts. No attempt will be made, however, to develop an axiomatic foundation of set theory. The interested reader can find detailed treatments on the foundation of set theory in references [8], [13], [17], and [20] in the bibliography at the end of this book.

The concept of a set plays an important role in every branch of modern mathematics. Although it seems easy, and natural, to define a set as a collection of objects, it has been shown in the past that this definition leads to contradictions. For this reason, in the foundation of set theory the notion of a set is left undefined (like the points and lines in geometry), and is described simply by its properties. In this book we shall mainly work with a number of specific "small" sets (like the Euclidean spaces $\mathbb{R}^n$ and their subsets), and we shall avoid making use of the "big" sets that lead to paradoxes. Therefore, a **set** is considered to be a collection of objects, viewed as a single entity.

Sets will be denoted by capital letters. The objects of a set A are called the **elements** (or the **members** or the **points**) of A. To designate that an object x belongs to a set A, the **membership symbol** $\in$ is used, that is, we write $x \in A$ and read it: x belongs to (or is a member of) A. Similarly, the symbolism $x \notin A$ means that the element x does not belong to A. Braces are also used to denote sets. For instance, the set whose elements are a, b, and c is written as $\{a, b, c\}$. A set having only one element is called a **singleton**.

Two sets A and B are said to be **equal**, in symbols $A = B$, if A and B have precisely the same elements. A set A is called a **subset** of (or that it is included in) a set B, in symbols $A \subseteq B$, if every element of A is also a member of B. Clearly, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$ both hold. If $A \subseteq B$ and $B \neq A$, then A is called a **proper subset** of B. The set without any elements is called the **empty** (or the **void**) set and is denoted by $\emptyset$. The empty set is a subset of every set.

If A and B are two sets, then we define

i. the **union** $A \cup B$ of A and B to be the set

$$A \cup B = \{x\colon\ x \in A \text{ or } x \in B\};$$

ii. the **intersection** $A \cap B$ of A and B to be the set

$$A \cap B = \{x\colon\ x \in A \text{ and } x \in B\};$$

iii. the set **difference** $A \setminus B$ of B from A to be the set

$$A \setminus B = \{x\colon\ x \in A \text{ and } x \notin B\}.$$

The set $A \setminus B$ is sometimes called the complement of B relative to A. Two sets A and B are called **disjoint** if $A \cap B = \emptyset$.

A number of useful relationships among sets are listed below, and the reader is expected to be able to prove them:

1. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$;
2. $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$;

3. $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$;
4. $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$.

The identities (1) and (2) between unions and intersections are referred to as the **distributive laws**.

We remind the reader how one goes about proving the preceding identities by showing (1). Note that an equality between two sets has to be established, and this shall be done by verifying that the two sets contain the same elements. Thus, the argument for (1) goes as follows:

$$x \in (A \cup B) \cap C \iff x \in A \cup B \text{ and } x \in C \iff (x \in A \text{ or } x \in B) \text{ and } x \in C$$
$$\iff x \in A \cap C \text{ or } x \in B \cap C \iff x \in (A \cap C) \cup (B \cap C).$$

Another useful concept is the symmetric difference of two sets. If A and B are sets, then their **symmetric difference** is defined to be the set

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

The concepts of union and intersection of two sets can be generalized to unions and intersections of arbitrary families of sets. A **family of sets** is a nonempty set $\mathcal{F}$ whose members are sets by themselves. There is a standard way for denoting a family of sets. If for each element i of a nonempty set I, a subset A_i of a fixed set X is assigned, then $\{A_i\}_{i \in I}$ (or $\{A_i: i \in I\}$ or simply $\{A_i\}$) denotes the family whose members are the sets A_i. The nonempty set I is called the **index set** of the family, and its members are known as indices. Conversely, if $\mathcal{F}$ is a family of sets, then by letting $I = \mathcal{F}$ and $A_i = i$ for each $i \in I$, we can express $\mathcal{F}$ in the form $\{A_i\}_{i \in I}$.

If $\{A_i\}_{i \in I}$ is a family of sets, then the **union** of the family is defined to be the set

$$\bigcup_{i \in I} A_i = \{x \colon \exists\, i \in I \text{ such that } x \in A_i\},$$

and the **intersection** of the family by

$$\bigcap_{i \in I} A_i = \{x \colon x \in A_i \text{ for each } i \in I\}.$$

Occasionally, $\bigcup_{i \in I} A_i$ will be denoted by $\bigcup A_i$ and $\bigcap_{i \in I} A_i$ by $\bigcap A_i$. Also, if $I = \mathbb{N} = \{1, 2, \ldots\}$ (the set of natural numbers), then the union and intersection of the family will be denoted by $\bigcup_{n=1}^{\infty} A_n$ and $\bigcap_{n=1}^{\infty} A_n$, respectively. The dummy index n can be replaced, of course, by any other letter.

The **distributive laws** for general families of sets now take the form

$$\Big(\bigcup_{i\in I} A_i\Big)\cap B = \bigcup_{i\in I}(A_i \cap B) \quad\text{and}\quad \Big(\bigcap_{i\in I} A_i\Big)\cup B = \bigcap_{i\in I}(A_i \cup B).$$

A family of sets $\{A_i\}_{i\in I}$ is called **pairwise disjoint** if for each pair i and j of distinct indices, the sets A_i and A_j are disjoint, i.e., $A_i \cap A_j = \emptyset$. The set of all subsets of a set A is called the **power set** of A, and is denoted by $\mathcal{P}(A)$. Note that $\emptyset$ and A are members of $\mathcal{P}(A)$. For most of our work in this book, subsets of a fixed set X will be considered (the set X can be thought of as a frame of reference), and all discussions will be considered with respect to the basic set X.

Now, let X be a fixed set. If $P(x)$ is a property (i.e., a well-defined "logical" sentence) involving the elements x of X, then the set of all x for which $P(x)$ is true will be denoted by $\{x \in X\colon\ P(x)\}$. For instance, if $X = \{1, 2, \dots\}$ and $P(x)$ represents the statement "The number $x \in X$ is divisible by 2," then $\{x \in X\colon\ P(x)\} = \{2, 4, 6, \dots\}$.

If A is a subset of X, then its **complement** A^c (relative to X) is the set $A^c = X \setminus A = \{x \in X\colon\ x \notin A\}$. It should be obvious that $(A^c)^c = A$, $A \cap A^c = \emptyset$, and $A \cup A^c = X$. Some other properties of the complement operation are stated next (where A and B are assumed to be subsets of X):

5. $A \setminus B = A \cap B^c$;
6. $A \subseteq B$ if and only if $B^c \subseteq A^c$;
7. $(A \cup B)^c = A^c \cap B^c$;
8. $(A \cap B)^c = A^c \cup B^c$.

The identities (7) and (8) are referred to as De Morgan's[1] laws. The generalized De Morgan's laws are going to be very useful, and for this reason we state them as a theorem.

Theorem 1.1 (De Morgan's Laws). *For a family $\{A_i\}_{i\in I}$ of subsets of a set X, the following identities hold:*

$$\Big(\bigcup_{i\in I} A_i\Big)^c = \bigcap_{i\in I} A_i^c \quad\text{and}\quad \Big(\bigcap_{i\in I} A_i\Big)^c = \bigcup_{i\in I} A_i^c.$$

[1] Augustin De Morgan (1806–1871), a British mathematician. He is well known for his contributions to mathematical logic.

Proof. We establish the validity of the first formula only, and we leave the verification of the other for the reader. Note that

$$x \in \Big(\bigcup_{i\in I} A_i\Big)^{c} \iff x \notin \bigcup_{i\in I} A_i \iff x \notin A_i \text{ for all } i \in I$$
$$\iff x \in A_i^{c} \text{ for all } i \in I \iff x \in \bigcap_{i\in I} A_i^{c},$$

and this establishes the first identity. ■

By a **function** f from a set A to a set B, in symbols $f\colon A \to B$ (or $A \xrightarrow{f} B$ or even $x \mapsto f(x)$), we mean a specific "rule" that assigns to each element x of A a unique element y in B. The element y is called the **value** of the function f at x (or the **image** of x under f) and is denoted by $f(x)$, that is, $y = f(x)$. The element $y = f(x)$ is also called the **output** of the function when the **input** is x. The set A is called the **domain** of f, and the set $\{y \in B\colon \exists\, x \in A \text{ with } y = f(x)\}$ is called the **range** of f. It is tacitly understood that the sets A and B are nonempty.

Two functions $f\colon A \to B$ and $g\colon A \to B$ are said to be **equal**, in symbols $f = g$, if $f(x) = g(x)$ holds true for each $x \in A$. A function $f\colon A \to B$ is called **onto** (or **surjective**) if the range of f is all of B; that is, if for every $y \in B$ there exists (at least one) $x \in A$ such that $y = f(x)$. The function $f\colon A \to B$ is called **one-to-one** (or **injective**) if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

Now, let $f\colon X \to Y$ be a function. If A is a subset of X, then the **image** $f(A)$ of A under f is the subset of Y defined by

$$f(A) = \{y \in Y\colon \exists\, x \in A \text{ such that } y = f(x)\}.$$

Similarly, if B is a subset of Y, then the **inverse image** $f^{-1}(B)$ of B under f is the subset of X defined by $f^{-1}(B) = \{x \in X\colon f(x) \in B\}$. Regarding images and inverse images of sets, the following relationships hold (we assume that $\{A_i\}_{i\in I}$ is a family of subsets of X and $\{B_i\}_{i\in I}$ a family of subsets of Y):

9. $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$;
10. $f(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} f(A_i)$;
11. $f^{-1}(\bigcup_{i\in I} B_i) = \bigcup_{i\in I} f^{-1}(B_i)$;
12. $f^{-1}(\bigcap_{i\in I} B_i) = \bigcap_{i\in I} f^{-1}(B_i)$;
13. $f^{-1}(B^{c}) = (f^{-1}(B))^{c}$.

Given two functions $f\colon X \to Y$ and $g\colon Y \to Z$, their **composition** $g \circ f$ is the function $g \circ f\colon X \to Z$ defined by $(g \circ f)(x) = g(f(x))$ for each $x \in X$.

If a function $f\colon X \to Y$ is one-to-one and onto, then for every $y \in Y$ there exists a unique $x \in X$ such that $y = f(x)$; the unique element x is denoted by $f^{-1}(y)$. Thus, in this case, a function $f^{-1}\colon Y \to X$ can be defined by $f^{-1}(y) = x$, whenever $f(x) = y$. The function f^{-1} is called **the inverse** of f. Note that $(f \circ f^{-1})(y) = y$ for all $y \in Y$ and $(f^{-1} \circ f)(x) = x$ for all $x \in X$. The latter relations are often written as $f \circ f^{-1} = I_Y$ and $f^{-1} \circ f = I_X$, where $I_X: X \to X$ and $I_Y: Y \to Y$ denote the identity functions; that is, $I_X(x) = x$ and $I_Y(y) = y$ for all $x \in X$ and $y \in Y$.

Any function $x: \mathbb{N} \to X$, where $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers, is called a **sequence** of X. The standard way to denote the value $x(n)$ is by x_n (called the n^{th} **term of the sequence**). We shall denote the sequence x by $\{x_n\}$, and we shall consider it both as a function and as a subset of X. A **subsequence** of a sequence $\{x_n\}$ is a sequence $\{y_n\}$ for which there exists a strictly increasing sequence $\{k_n\}$ of natural numbers (that is, $1 \le k_1 < k_2 < k_3 < \cdots$) such that $y_n = x_{k_n}$ holds for each n.

If now $\{A_i\}_{i \in I}$ is a family of sets, then the **Cartesian**[2] **product** $\Pi_{i \in I} A_i$ (or ΠA_i) is defined to be the set consisting of all functions $f: I \to \bigcup_{i \in I} A_i$ such that $x_i = f(i) \in A_i$ for each $i \in I$. Such a function is called (for obvious reasons) a **choice function** and quite often is denoted by $(x_i)_{i \in I}$ or simply by (x_i).

If a family of sets consists of two sets, say A and B, then the Cartesian product of the sets A and B is designated by $A \times B$. The members of $A \times B$ are denoted as **ordered pairs**, that is,

$$A \times B = \{(a, b): \ a \in A \ \text{ and } \ b \in B\}.$$

Clearly, $(a, b) = (a_1, b_1)$ if and only if $a = a_1$ and $b = b_1$. Similarly, the Cartesian product of a finite family of sets $\{A_1, \ldots, A_n\}$ is written as $A_1 \times \cdots \times A_n$ and its members are denoted as n-tuples, that is,

$$A_1 \times \cdots \times A_n = \{(a_1, \ldots, a_n): \ a_i \in A_i \ \text{ for each } \ i = 1, \ldots, n\}.$$

Here, again $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ if and only if $a_i = b_i$ for $i = 1, \ldots, n$. If $A_1 = A_2 = \cdots = A_n = A$, then it is standard to write $A_1 \times \cdots \times A_n$ as A^n. Similarly, if the family of sets $\{A_i\}_{i \in I}$ satisfies $A_i = A$ for each $i \in I$, then $\Pi_{i \in I} A_i$ is written as A^I, that is, $A^I = \{f \mid f: I \to A\}$.

- *When is the Cartesian product of a family of sets $\{A_i\}_{i \in I}$ nonempty?*

Clearly, if the Cartesian product is nonempty, then each A_i must be nonempty. The following question may, therefore, be asked:

- *If each A_i is nonempty, is then the Cartesian product ΠA_i nonempty?*

Although the answer seems to be affirmative, it is unfortunate that such a statement cannot be proven with the usual axioms of set theory. The affirmative answer

[2] René Descartes or Cartesius (1596–1650), an influential French philosopher and mathematician. He is the founder of analytic geometry.