CONNECTIONS, SPRAYS AND FINSLER STRUCTURES

József Szilasi
Rezső L Lovas
Dávid Cs Kertész

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Rezső L Lovas
Dávid Cs Kertész
University of Debrecen, Hungary

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IN MEMORIAM

Ludwig Berwald

1883–1942

András Rapcsák

1914–1993

Makoto Matsumoto

1920–2005

Preface

The Tao that can be told is not the eternal Tao.
The name that can be named is not the eternal name.
The nameless is the beginning of heaven and earth.
The named is the mother of ten thousand things.

Lao Tsu (Tao Te Ching [65], Ch. 1)

Mathematics is the music of science and real analysis is the Bach of mathematics. There are many foolish things I could say about the subject of this book, but the foregoing will give the reader an idea of where my heart lies.

Sterling K. Berberian [11]

1. The three concepts in the title lie at the heart of our book. The following diagram shows that the order in which they appear is also important:

more special $\longrightarrow$

manifolds with connections	$\supset$	manifolds with sprays	$\supset$	manifolds with Finsler structures

$\longleftarrow$ *more general*

We note already at this point that by a connection we always mean a so-called *Ehresmann connection.* As a first approach, specifying an Ehresmann connection means the fixing of a direct summand of the (canonical) vertical subbundle of the double tangent bundle. Thus, it is a geometric notion, and indeed a very simple one.

Our main goal was to give a comprehensive introduction to the theory of *Finsler manifolds*, i.e., manifolds endowed with a Finsler structure or Finsler function. The scheme above shows that the appropriate approach is the study of manifolds endowed with connections and sprays. Above all, we had in mind the needs of PhD students in Finsler geometry, and we tried to summarize the fundamentals of differential geometry together with a reasonable amount of the prerequisites from algebra and analysis in a single volume and in a coherent manner, and to expose the rudiments of Finsler geometry on these foundations. Experienced readers will notice that this approach makes it possible to derive several classical theorems from general principles in a simple and unified way.

The table of contents comprises all the topics included in the book. In the following we only point out a few important features of our method.

2. As we have already mentioned, the first aim of our book is to serve as a textbook. To illustrate this, we note the following features:

(1) In principle, the reading of the book requires only the knowledge of undergraduate linear algebra and analysis. However, readers with a background in classical differential geometry or elementary Riemannian geometry will assimilate the material and perceive the subtleties more easily.

(2) We define our technical terms unambiguously and use them in this spirit throughout. We state all our assertions in a clear and explicit manner, and we give detailed proofs for them. There are only a few exceptions to the latter, but in these few cases we nearly always give explicit references to the literature. In our proofs we carefully explain all steps, and we never leave non-trivial details to the reader (or at least we do not intend to do so).

(3) A course in 'Advanced Calculus' is an organic part of the book. Such a course could be formed from the relevant parts, arranged logically as follows: *Appendix A, Chapter 1, Appendix B, Appendix C.1, Subsections 4.2.1, 3.2.1, Section 9.1.* A non-standard feature of this course is the study of Finsler vector spaces, which is in close connection with convexity. Our teaching experience shows that the interest of upper undergraduate students in Finsler geometry may be aroused in this way.

We note that Appendix C.2, C.3 and Subsection 6.1.10 can easily be made independent of manifold theory, perhaps with some tutorial help. In this way we obtain an introduction to the classical theory of hypersurfaces of a finite-dimensional real vector space.

3. The conceptual framework for the study of sprays and Finsler structures is provided by the theory of *manifolds* and *vector bundles*. It must be clear from the foregoing that we do not explicitly rely on the reader's knowledge of manifold theory. However, our discussion of manifolds is rather concise, we usually restrict ourselves to the presentation of the most important concepts and facts, and we omit difficult proofs. We do so not only due to the limited size of the book, but also because there are plenty of excellent textbooks on manifold theory, e.g., Barden and Thomas [9], Jeffrey M. Lee [66], John M. Lee [69], Michor [76] and Tu [96], just to mention a few recent ones. There are also good course materials available on the Internet. Our main guides were such classical works as the first volume of the monograph of Greub, Halperin and Vanstone [52] and 'Semi-Riemannian Geometry' by O'Neill [79].

We only need a few special types of vector bundles in this book. First of all, of course,

$$\text{the } \textit{tangent bundle } \tau\colon TM \to M$$

of a manifold M, and also some vector bundles over the tangent manifold TM:

$$\tau_{TM}\colon TTM \to TM, \quad \tau_*\colon TTM \to TM,$$
$$\tau^{\mathsf{v}}_{TM}\colon TTM \to TM \text{ – the } \textit{vertical subbundle},$$
$$\pi\colon TM \times_M TM \to TM \text{ – the } \textit{Finsler bundle}.$$

However, for the sake of transparency, and to avoid repetition, we introduce the basic concepts connected with bundles at the level of generality of fibre bundles and 'abstract vector bundles'. Even a part of the theory of Ehresmann connections might be discussed in such generality, but we think that this theory is easier to understand in the framework of the tangent bundle, and this approach also makes it possible to expound Ehresmann connections in their full complexity.

We introduce *covariant derivatives* at the general level of vector bundles. The reason is that we need them on different vector bundles, and thus the common definition ensures greater clarity.

The covariant derivatives on a manifold (more precisely, on its tangent bundle) are in a bijective correspondence with a special class of Ehresmann connections, the *linear Ehresmann connections.* Thus, these two objects show the two sides of the same coin: Ehresmann connections have a purely geometric character, whereas covariant derivatives act as differential operators. We discuss this fundamental relationship between covariant derivatives and linear Ehresmann connections in detail.

4. The reader will see that Ehresmann connections are indeed the cornerstones of the monumental edifice of differential geometry. Moreover, they also make it possible to give a visualizable geometric formulation and proof of several theorems in classical analysis and differential geometry.

The appearance of *sprays* on the scene makes the picture even more complex. Informally, a spray defines a special class of second-order differential equations over a manifold. The 'spray coefficients', which actually depend on the chart we are working with, can be interpreted as the equivalents of forces in Newtonian mechanics. Each homogeneous Ehresmann connection determines a spray in a natural manner. On the other hand, a homogeneous Ehresmann connection comes from a spray if, and only if, its torsion vanishes. Sprays provide a framework for a unified and systematic discussion of geodesics.

Now we are in a position to complete the explanation of the scheme sketched in paragraph 1. Each Finsler function gives rise to a spray in a canonical manner, which carries much information about the original structure. Thus an essential part of Finsler geometry may be discussed in the setting of spray geometry. The converse question of the 'Finsler metrizability' of sprays, i.e., when does a given spray come from a Finsler function as its canonical spray, leads to a jungle of difficult problems.

5. Besides the carefully chosen conceptual background, we also need a substantial and flexible technical apparatus. The major part of this is developed in Chapters 3, 4 and 6. We apply traditional tensor calculus, tensor derivations (covariant derivatives in particular), the classical graded derivations of a Grassmann algebra (Lie derivatives, substitution operators and exterior derivative) and the graded derivations induced by vector-valued forms and described by Frölicher – Nijenhuis theory. The canonical objects and constructions on the tangent bundle and on the Finsler bundle (canonical involution, vertical endomorphism, vertical and complete lifts, etc.) may also be regarded as parts of our technical apparatus. Finally, the theorems about existence, uniqueness and smooth dependence of solutions of ordinary differential equations also belong to our indispensable tools.

6. The heart of our book is made up of Chapters 5, 7, 8 and 9. We discuss Ehresmann connections and sprays in such a systematic and detailed manner that the relevant chapters, together with Chapter 4, can almost be considered as a monograph embedded in the textbook. The same cannot be said about the chapter discussing Finsler structures. This theory has grown so vast in the last two decades and it is still developing so intensely that we are able to present only some of the most important aspects of

Finsler manifolds.

To follow through the path

$$\text{connections} \to \text{sprays} \to \text{Finsler structures}$$

the reader has to climb a lot of mountains. That is what we imply by quoting the Korean proverb at the beginning of Section 3.4. We think, however, that the reaching of each peak will give the reader its own reward.

7. Hewitt and Stromberg wrote the following in the preface of their book 'Real and Abstract Analysis': 'exercises are to mathematicians what Czerny is to a pianist'. Although we completely agree with them, the reader will not find any explicit exercise in our book. The reason is that many of our assertions may be regarded as 'exercises', which are immediately followed by 'solutions', in the form of proofs. These assertions and their proofs usually have a technical character, which, of course, does not mean lack of quality. Most of them are easily recognizable, and we recommend ambitious readers to try to prove these assertions for themselves. If they find simpler or more elegant solutions than ours, then they are on the best way to profound understanding.

We try to apply the index-free or 'intrinsic' method in most of our proofs. Doing these proofs with the method of classical tensor calculus after choosing a chart may supply the reader with further useful exercises. (This way will often prove shorter.) Sometimes we only give a proof which uses coordinates. In these cases the reader may try to find an index-free argument.

We give very few numerical examples, since a large number of these would make the book too long, moreover, the most interesting and important examples are easily accessible. Here we refer to Z. Shen's monograph [88].

For the readers' convenience, errata and addenda will be available on the first author's homepage (http://www.math.unideb.hu/~szilasi) as soon as we find it necessary.

J. Szilasi
R. L. Lovas
D. Cs. Kertész

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