

Third Edition



Number Treasury³

Investigations, Facts and Conjectures about
More than 100 Number Families

Margaret J Kenney • Stanley J Bezuska

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Boston College, Massachusetts, USA



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Number Treasury³ is dedicated to my co-author Stanley J. Bezuska, longtime Professor of Mathematics and Director of the Mathematics Institute at Boston College, who was a source of inspiration and encouragement to so many in the world of mathematics education.

Margaret J. Kenney

Foreword

The gift of numbers like the gift of fire has made the world much brighter.

Stanley J. Bezuska (1914–2008)

Introduction

Time and numbers were born together. Time is a measure of change, and numbers express that measure. There is an awesome mystery that surrounds the all-pervasive dimension of time. Likewise, there is an awesome mystery and fascination about numbers that attract distinguished mathematics researchers as well as imaginative amateurs the world over. Teachers who pursue numbers and the rich history of numbers with their classes can provide students with an understanding that mathematics is a collaborative effort that has been nurtured by individuals and groups representing many cultures and periods of history. Students can learn to become accomplished investigators, to make discoveries, and contribute to a branch of mathematics that is vibrant and motivating. **Number Treasury**³ has evolved in order to serve as a catalyst for those who ascribe to this point of view.

Details

Number Treasury³ is a broadening and update of **Number Treasury**². The book contains information about more than 100 families of positive integers. Brief historical notes often accompany the descriptions and examples of the number families. Exercises for each major family are provided to stimulate insight. Some exercises contain problems that are thought provokers to be resolved simply with paper and pencil; others should be tackled with calculator in hand so that lengthier computations can be managed with ease and take the results to a higher level of understanding. Still other problems are intended for more extensive exploration with the use of computer software. In some instances it is helpful to model problems with hands-on materials.

The emphasis in **Number Treasury**³ is on doing rather than proving. However, the reader is urged to think critically about situations, to provide reasoned explanations, to make generalizations and to formulate conjectures. The book begins with a chapter of Investigations. These are principally stand-alone activities that represent content drawn from the Chapters 2 through 7 of the book. Their purpose is to set the tone of the book and to stimulate student reflection and research in a variety of areas. In fact, throughout the book, the reader will find numerous open-ended problems. This book also contains detailed solutions to the Exercises and Investigations. A Glossary and Index are provided for quick access to information. References and recommended readings are supplied so that teachers and students can use this book as a stepping stone to more concentrated study.

Who Uses Number Treasury³

This book is written for teachers and students. For teachers **Number Treasury**³ is a resource for instructional preparation and problems, together with snapshots of mathematical history intended for teachable moments. For students who are engaged in learning about number families and who are assigned problems, projects and papers, **Number Treasury**³ is a useful source of ideas and topics. The mix of discussion with examples and illustrations is intended to serve as a writing model for the student. Both audiences should think critically about the content, provide carefully reasoned explanations, make generalizations, and form conjectures.

Who Is Involved

The first edition was completed with the able assistance of six Boston College graduates and undergraduates: Jeanne Cavanaugh, James Cavanaugh, Claudia Katze, Stephen Kokoska, Jill Nille, and Jonathan Smith.

Seven Boston College graduates and undergraduates were indispensable in the production of the second edition. Special thanks and grateful appreciation go to Joan Martin for her thoughtful content and style suggestions, editorial advice and word processing skills; to Cynthia Tahlmore, Geraldine Mele, and Erin Mitchell for computer graphics and word processing assistance; to Allyson Russo, Shannon Toomey, and Megan Mazzara for problem solutions.

The third edition has been completed by the surviving original author with the invaluable assistance and perseverance of Geraldine Mele who offered not only content suggestions but who also especially contributed word processing, computer graphics and style expertise. Sincere gratitude and appreciation is also extended to Joan Martin for her careful review of the manuscript.

Contents

<i>Foreword</i>	xi
1. A Perfect Number of Investigations $28 = 1 + 2 + 4 + 7 + 14$	1
2. Numbers Based on Divisors and Proper Divisors	39
Positive Integers	39
Divisors, Multiples and Proper Divisors	39
Prime and Composite Numbers	40
Sieve of Eratosthenes	41
Prime Factorization Property (Fundamental Theorem of Arithmetic)	42
Testing for Primes	44
Divisors of an Integer, GCD, and LCM	45
Relatively Prime and Euler ϕ Numbers	46
Abundant, Deficient, and Perfect Numbers	48
Sums and Differences of Abundant and Deficient Numbers	49
Products of Abundant and Deficient Numbers	53
Multiples of Perfect Numbers	54
Consecutive Integers and Abundant Numbers	55
Abundant Numbers as Sums of Abundant Numbers	56
Powers of Primes and Deficient Numbers	56
Even and Odd Integers, Even Perfect Numbers, Mersenne Primes	58
Multiply Perfect Numbers	61
Almost Perfect Numbers	62
Semiperfect Numbers	62
Weird Abundant Numbers	63
Operations on Semiperfect Numbers	64
Primitive Semiperfect Numbers	64
Amicable Numbers	65
Imperfectly Amicable Numbers	66
Sociable Numbers and Crowds	67

Practical Numbers	68
Baselike Numbers	71
3. Plane Figurate Numbers	73
Polygons	73
Figurate Numbers	74
Triangular Numbers	74
Operations on Triangular Numbers	77
Perfect Numbers, Triangular Numbers and Sums of Cubes	78
Pascal's Triangle	79
Triangle Inequality Numbers	80
Rectangular Numbers	83
Square Numbers	85
Sums of Square Numbers	88
Positive Square Pair Numbers	89
Bigrade Numbers	90
Pythagorean Triples	90
Primitive Pythagorean Triples	92
Congruent Numbers	93
Fermat's Last Theorem	94
Happy Numbers	94
Operations on Happy Numbers	95
Happy Number Words	96
Repeating Cycles	97
Patterns in Squares of 1, 11, 111,	98
Squarefree Numbers	99
Tetragonal Numbers	100
Pentagonal Numbers	101
Hexagonal Numbers	103
Recursion and Figurate Numbers	107
Remainder Patterns in Figurate Numbers	109
Gnomic Numbers	109
Lo-Shu Magic Square; Male and Female Numbers	111
4. Solid Figurate Numbers	113
Polyhedra and Solid Figurate Numbers	113
Pyramidal Numbers	115
Tetrahedral Numbers, Triangular Pyramidal Numbers	115
Square Pyramidal Numbers	118
Pentagonal Pyramidal Numbers	120
Hexagonal Pyramidal Numbers	120
Heptagonal and Octagonal Pyramidal Numbers	121

Star Numbers and Star Pyramidal Numbers	121
Rectangular Pyramidal Numbers	122
Cubic Numbers	125
Integers as Sums and Differences of Cubic Numbers, 1729	126
Pythagorean Parallelepiped Numbers	128

5. More Prime Connections 131

Goldbach's Conjectures	131
Integers as Sums of Odd Integers	133
Integers as Sums of Two Composite Numbers	134
Positive Prime Pair Numbers	134
Prime Line and Prime Circle Numbers	136
Beprisque Numbers	138
A Primes-Between Property	138
Germain Primes	139
Twin Primes	140
Semiprimes and Boolean Integers	140
Snowball Primes	142
Lucky Numbers	142
Prime and Lucky Numbers	144
Polya's Conjecture about Odd- and Even-Type Integers	145
Balanced Numbers	147
Fermat Numbers	147
Cullen Numbers	148
Ruth–Aaron Numbers	149

6. Digital Patterns and Noteworthy Numbers 151

Monodigit and Repunit Numbers and Langford Sequences	151
Social and Lonely Numbers	153
Additive Multidigital Numbers	155
Multiplicative Multidigital Numbers	156
Kaprekar's Number 6174, 99 and 1089	157
Doubling Numbers	159
Good Numbers	161
Nearly Good Semiperfect Numbers	163
Powerful Numbers	164
Armstrong Numbers and Digital Invariant Numbers	165
Narcissistic Numbers	167
Additive Digital Root Numbers	167
Additive Persistence of Integers	169
Multiplicative Digital Root Numbers	170
Multiplicative Persistence of Integers	171

Modest and Extremely Modest Numbers	173
Visible Factor Numbers	174
Nude Numbers	175
7. More Patterns and Other Interesting Numbers	177
More Sums of Consecutive Integers	177
Product Patterns for Consecutive Integers	181
Consecutive Integer Divisors	183
Consecutive Number Sums and Square Numbers	184
Factorial Numbers, Applications and Extensions	185
Factorial Sum Numbers and Subfactorial Numbers	188
Hailstone and Ulam Numbers; The Collatz and Ulam Conjecture	189
Palindromic Numbers	191
Creating Palindromic Numbers	193
Palindromic Number Words and Curiosities	194
Palindromic Numbers and Figurate Numbers	196
Palindromic Primes and Emirps	196
Honest Numbers	197
Bell Numbers	198
Catalan Numbers	200
Fibonacci Numbers	203
Lucas Numbers	206
Tribonacci Numbers	207
Tetranacci Numbers	208
Phibonacci Numbers	209
Survivor Numbers or U-Numbers or Ulam Numbers	209
Tautonymic Numbers	210
Lagado Numbers	211
<i>Recommended Readings</i>	215
<i>Glossary of Numbers</i>	217
<i>Solutions to Investigations</i>	225
<i>Solutions to Exercises</i>	249
<i>Index</i>	305

Chapter 1

A Perfect Number of Investigations

$$28 = 1 + 2 + 4 + 7 + 14$$

A GREAT discovery solves a great problem but there is a grain of discovery in any problem.
George Polya (1887–1985)

What are They?

The Investigations that follow are a set of stand-alone activities. Each Investigation focuses on at least one number family or topic relating to numbers. All but six of the Investigations are described on one page. Share with students that the problems in an Investigation are intended to be challenging in many ways:

- The time needed to complete an Investigation may vary and exceed the time required to finish a typical homework assignment.
- The computation necessary to bring closure may be lengthy and demanding — even with the use of technology.
- The amount of writing, discussing, explaining and illustrating may be more than anticipated.

Teacher Tips

The Investigations are listed in ascending order of difficulty in the table on the next two pages. There are three levels of difficulty represented in the 28 Investigations that can be assigned individually or adapted for group work. The lowest level consists of the first eight Investigations that have the least prerequisites. The middle level consists of the next nine Investigations and requires more use of abstract reasoning and familiarity with algebraic expressions. The final 11 Investigations challenge the student to persist and probe more deeply in order to complete the work.

There is also a column in the table naming the most significant prerequisite(s) needed by the student to understand and carry out the work in each Investigation. The teacher may choose to provide additional content background for some specific Investigations prior to assigning them. Assign the Investigations as extended homework or as in-class work. Some Investigations call for the preparation of

reports. Thus, students may need further directions, especially about the kind of resources available for them to use. Students should know the Internet is an excellent resource, and that it should be used appropriately as they compile their reports.

Finally, pages noted in the Prerequisites column refer to related material contained in Chapters 2 through 7.

Page	Investigation	Prerequisites & Text reference
4	Footsteps of Lagrange	Square numbers, p. 85
5,6	Trying Trapezoids	Triangular numbers & trapezoids, p. 74
7	Hexagons in Black & White	Familiarity with recursive & explicit formulas, p. 107
8,9	Marble Art	Figurate numbers, following directions, pp. 74–111
10	Honest Number Hunt	Counting in a language, searching resources, p. 197
11	Seeking Honesty in Numbers	Groups work together to organize their data, p. 197
12	Geoboard Journeys	Trial/Error pursuit, link Catalan & Pascal, p. 200
13	Mysterious Mountains & Binary Trees	Doing & arranging sketches, p. 200
14	Fermat Factorings	Prime factorization, p. 42
15,16	Factor Lattices	LCM, prime factorization, p. 46
17	A Juggling Act	Describing systematically, p. 46
18	The Super Sum	Reasoning with patterns, p. 74
19,20	Conjecturing with Pascal	Articulating patterns, p. 79
21	Pythagorean Triple Pursuits	Evaluating expressions, p. 90
22	Pentagonal Play	Squares & triangles within, p. 101
23	Triangular Number Turnarounds	Visualizing triangles, p. 80
24	Centered Triangular Numbers	Making algebraic generalizations, p. 74
25,26	Catalan Capers	Connect algebra & geometry, p. 201
27	Highly Composite Numbers	Counting divisors, p. 45
28,29	Tower of Hanoi & the Reve's Puzzle	Recursive actions & thinking, p. 56
30	Perfect Number Patterns	Using logs to count, p. 59
31	Crisscross Cubes	Perimeter, area, volume, make generalizations, p. 125

Page	Investigation	Prerequisites & Text reference
32	A Medieval Pattern	Connect pattern to shape & number, p. 88
33	Prime Magic	Trial & error yields results, p. 136
34	Dealing with Digits (base ten)	Exploring ways to count, p. 188
35	Factorial Finishes	Importance of 2×5 in reasoning, p. 185
36	Designing Designs	Representation is critical, p. 198
37	Fibonacci Fascinations	Spreadsheet use, p. 203

Footsteps of Lagrange

Develop the first 20 terms of the sequence s_n , where s_n is the number of ways n can be written as a sum of at most 4 squares. Note one term will be counted as a sum.

EXAMPLE

How many ways can 5 be written as a sum of at most 4 squares?
1 way since $5 = 4 + 1 = 2^2 + 1^2$.

EXAMPLE

How many ways can 9 be written as a sum of at most 4 squares?
2 ways since $9 = 2^2 + 2^2 + 1^2$ and $9 = 3^2$.

1. Fill in the following table.

n	Ways to Write n as Sum of at Most 4 Squares	Number of Ways
1		
2		
3		
4		
5	$2^2 + 1^2$	1
6		
7		
8		
9	$3^2; 2^2 + 2^2 + 1^2$	2
10		
11		
12		
13		
14		
15		
16		
17		
18		
19		
20		

2. Describe in a few sentences some patterns you observe in the table.
3. Find a number and verify that it can be written as a sum of at most 4 squares in exactly

a) three ways

b) four ways