国外数学名著系列

（影印版）14

Petter Bjørstad Mitchell Luskin (Eds.)

Parallel Solution of Partial Differential Equations

偏微分方程的并行算法

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格（Springer）出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

FOREWORD

This IMA Volume in Mathematics and its Applications

PARALLEL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

is based on the proceedings of a workshop with the same title. The workshop was an integral part of the 1996–97 IMA program on "MATHEMATICS IN HIGH-PERFORMANCE COMPUTING."

I would like to thank Petter Bjørstad of the Institutt for Informatikk, University of Bergen and Mitchell Luskin of the School of Mathematics, University of Minnesota for their excellent work as organizers of the meeting and for editing the proceedings.

I also take this opportunity to thank the National Science Foundation (NSF), Department of Energy (DOE), and the Army Research Office (ARO), whose financial support made the workshop possible.

Willard Miller, Jr., Professor and Director

PREFACE

The numerical solution of partial differential equations has been of major importance to the development of many technologies and has been the target of much of the development of parallel computer hardware and software. Parallel computers offer the promise of greatly increased performance and the routine calculation of previously intractable problems. The papers in this volume were presented at the IMA workshop on the Parallel Solution of PDE held during June 9–13, 1997. The workshop brought together leading numerical analysts, computer scientists, and engineers to assess the state-of-the-art and to consider future directions.

This volume contains papers on the development and assessment of new approximation and solution techniques that can take advantage of parallel computers. Topics include domain decomposition methods, parallel multi-grid methods, front tracking methods, sparse matrix techniques, adaptive methods, fictious domain methods, and novel time and space discretizations. Applications discussed include fluid dynamics, radiative transfer, solid mechanics, and semiconductor simulation.

We would like to thank the IMA for giving us the opportunity to hold this workshop and the National Science Foundation (NSF), the Department of Energy (DOE), and the Army Research Office (ARO) for providing financial support. Individual thanks are extended to Avner Friedman and Robert Gulliver of the IMA for coordinating, scheduling, and providing logistic support for the workshop and to Patricia V. Brick and Dzung N. Nguyen of the IMA for providing editorial support.

Petter Bjørstad, Bergen, Norway

Mitchell Luskin, Minneapolis, Minnesota

CONTENTS

ITERATIVE SUBSTRUCTURING METHODS FOR SPECTRAL ELEMENT DISCRETIZATIONS OF ELLIPTIC SYSTEMS IN THREE DIMENSIONS

LUCA F. PAVARINO* AND OLOF B. WIDLUND†

Abstract. Spectral element methods are considered for symmetric elliptic systems of second-order partial differential equations, such as the linear elasticity and the Stokes systems in three dimensions. The resulting discrete problems can be positive definite, as in the case of compressible elasticity in pure displacement form, or saddle point problems, as in the case of almost incompressible elasticity in mixed form and Stokes equations. Iterative substructuring algorithms are developed for both cases. They are domain decomposition preconditioners constructed from local solvers for the interior of each element and for each face of the elements and a coarse, global solver related to the wire basket of the elements. In the positive definite case, the condition number of the resulting preconditioned operator is independent of the number of spectral elements and grows at most in proportion to the square of the logarithm of the spectral degree. For saddle point problems, there is an additional factor in the estimate of the condition number, namely, the inverse of the discrete inf-sup constant of the problem.

Key words. linear elasticity, Stokes problem, spectral element methods, mixed methods, preconditioned iterative methods, substructuring, Gauss-Lobatto-Legendre quadrature.

AMS(MOS) subject classifications. 65N30, 65N35, 65N55.

1. Introduction. The goal of this paper is to formulate and study iterative substructuring methods for symmetric elliptic systems of second-order partial differential equations in three dimensions. Important examples, which are considered in some detail, are the equations of linear elasticity and Stokes. We consider conforming spectral finite element discretizations based on a Galerkin formulation of the problem and Gauss-Lobatto-Legendre quadrature. The resulting discrete systems are either positive definite, as in the case of compressible elasticity in pure displacement form, or of saddle point form, as in the case of almost incompressible elasticity in mixed form and Stokes problems. For these three cases, we introduce iterative substructuring algorithms which extends our earlier work

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*Department of Mathematics, University of Milano, Via Saldini 50, 20133 Milano, ITALY (pavarino@ares.mat.unimi.it). URL: http://ipv512.unipv.it/~webmat2/pavarino/pavarino.html. This work was supported by I.A.N-CNR, Pavia and by the National Science Foundation under Grant NSF-CCR-9503408. Work on this project began when both of the authors were in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23681-0001 and were supported by NASA Contract No. NAS1-19480.

†Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, N.Y. 10012 (widlund@cs.nyu.edu). URL: http://cs.nyu.edu/cs/faculty/widlund/index.html. This work was supported in part by the National Science Foundation under Grants NSF-CCR-9503408 and in part by the U.S. Department of Energy under contract DE-FG02-92ER25127.

[30, 33] on scalar second-order elliptic equations. We recall that iterative substructuring methods are domain decomposition algorithms, in which we, implicitly, solve a reduced Schur complement system that is obtained by eliminating the variables interior to all the subregions into which the given region has been divided; cf., e.g., Smith, Bjørstad, and Gropp [38] or Dryja, Smith, and Widlund [12]. We consider iterative substructuring methods of wire basket type, where a preconditioner for the Schur complement is built from local solvers for each face (shared by two elements) and a coarse solver related to the wire basket (the union of the edges and the vertices of the elements). The main result for positive definite problems is a bound on the condition number of the preconditioned operator, which is independent of the number of spectral elements and is bounded from above by the square of the logarithm of the spectral degree. For saddle point problems, the reduced Schur complement is itself a saddle point problem, involving the interface unknowns and piecewise constant Lagrange multipliers. We then use a Krylov space method with a block-diagonal or block-triangular preconditioner using our wire basket preconditioner for the interface block. The main result for saddle point problems is a bound on the condition numbers of the preconditioned operators, which in this case is the product of a polylogarithmic factor and the inverse of inf-sup constant of the problem. Proofs of our results and additional details will be presented in two articles; see [31, 32].

We note that other iterative substructuring methods have been proposed in recent years. For positive definite systems, see, e.g., Mandel [26, 27], Le Tallec [22], and Farhat and Roux [16] and for saddle point problems, see Bramble and Pasciak [5], Quarteroni [34], Fischer and Rønquist [17], Maday, Meiron, Patera, and Rønquist [24], Rønquist [35], Le Tallec and Patra [23], and Casarin [10]. We also note that alternative iterative methods have been considered for saddle point problems, such as Uzawa's algorithm, multigrid methods, block-diagonal and block-triangular preconditioners; see, e.g., Elman [13, 14], Brenner [6], Klawonn [20], and the references therein.

The rest of the paper is organized as follows. In Section 2, we introduce the three elliptic systems which will serve as model problems throughout the paper: compressible linear elasticity in pure displacement form, incompressible and almost incompressible linear elasticity in mixed form, and the Stokes system. In Section 3, the spectral element discretization of these systems and GLL quadrature are described briefly. In Section 4, we introduce some extension operators from the interface that are needed in the construction of our preconditioners: the discrete harmonic, elastic, Stokes and mixed elastic extensions. An additional extension operator associated with the wire basket is also introduced. In Section 5, we describe our wire basket preconditioner for positive definite systems, both in matrix and variational form, and formulate the main result on the condition number of the preconditioned operator. In Section 6, we turn our attention to saddle point problems, starting with the description of the basic substructuring technique the use of which leads to a saddle point Schur complement. We then study the stability of this Schur complement problem and introduce block preconditioners built on our wire basket preconditioner for the positive definite case. Our main results for both the Stokes and and the incompressible elasticity problems are also formulated. Section 7 concludes the paper with results of some of our numerical experiments for problems in three dimensions.

2. Model elliptic systems. In this section, we will introduce three symmetric elliptic systems: compressible linear elasticity in pure displacement form, incompressible and almost incompressible linear elasticity in mixed form, and the Stokes system. The first is coercive, while the other two provide examples of saddle point problems. We will work with spectral element discretizations of these systems and introduce and study iterative substructuring methods for these concrete cases. However, the same techniques can be applied to other well-posed symmetric elliptic systems as well.

Throughout the paper, we will denote vector quantities by bold face characters.

2.1. Compressible linear elasticity in pure displacement form. Let $\Omega \subset R^3$ be a polyhedral domain, let Γ_0 be a nonempty subset of its boundary, and let $\mathbf{V}$ be the Sobolev space $\mathbf{V} = \{\mathbf{v} \in H^1(\Omega)^3 : \mathbf{v}|_{\Gamma_0} = 0\}$. The linear elasticity problem consists in finding the displacement $\mathbf{u} \in \mathbf{V}$ of the domain Ω, fixed along Γ_0, resulting from a surface force of density $\mathbf{g}$, along $\Gamma_1 = \partial\Omega - \Gamma_0$, and a body force $\mathbf{f}$:

$$
\begin{aligned}
a(\mathbf{u},\mathbf{v}) &= 2\mu \textstyle\int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v})\ dx + \lambda \int_\Omega \operatorname{div}\mathbf{u}\ \operatorname{div}\mathbf{v}\ dx \\
&= <\mathbf{F},\mathbf{v}> \quad \forall \mathbf{v} \in \mathbf{V}.
\end{aligned} \tag{1}
$$

Here λ and μ are the Lamé constants, $\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ the linearized strain tensor, and the inner products are defined as

$$
\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) = \sum_{i=1}^{3}\sum_{j=1}^{3} \epsilon_{ij}(\mathbf{u})\epsilon_{ij}(\mathbf{v}),
$$

$$
<\mathbf{F},\mathbf{v}> \ = \int_\Omega \sum_{i=1}^{3} f_i v_i\ dx + \int_{\Gamma_1} \sum_{i=1}^{3} g_i v_i\ ds.
$$

This pure displacement model is a good formulation for compressible materials, for which the Poisson ratio $\nu = \dfrac{\lambda}{2(\lambda+\mu)}$ is strictly less than 1/2, e.g., $\nu \le 0.4$; see, e.g., Ciarlet [11] for a detailed treatment of nonlinear and linear elasticity.

2.2. Almost incompressible linear elasticity in mixed form. When λ approaches infinity, the pure displacement model describes materials that are almost incompressible. In terms of the Poisson ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$, such materials are characterized by values of ν close to 1/2. It is well-known that when low order, h-version finite elements are used in the discretization of (1), *locking* can cause a severe deterioration of the convergence rate as $h \to 0$; see, e.g., Babuška and Suri [1]. If the p-version is used instead, locking in $\mathbf{u}$ is eliminated, but it could still occur in quantities of interest such as $\lambda \mathrm{div}\mathbf{u}$. Moreover, the stiffness matrix obtained by discretizing the pure displacement model (1) has a condition number that goes to infinity when $\nu \to 1/2$. Therefore, the convergence rate of any iterative method must also be expected to deteriorate rapidly as the material becomes almost incompressible.

Locking can be eliminated by introducing a space of Lagrange multipliers $U = L^2(\Omega)$ and the new variable $p = -\lambda \mathrm{div}\mathbf{u} \in U$ and by replacing the pure displacement problem with a mixed formulation:
Find $(\mathbf{u}, p) \in \mathbf{V} \times U$ such that

$$(2) \quad \begin{cases} 2\mu \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) \, dx - \int_\Omega \mathrm{div}\mathbf{v} \; p \, dx & = < \mathbf{F}, \mathbf{v} > \quad \forall \mathbf{v} \in \mathbf{V} \\ -\int_\Omega \mathrm{div}\mathbf{u} \; q \, dx - \frac{1}{\lambda} \int_\Omega pq \, dx & = \quad 0 \quad\quad \forall q \in U; \end{cases}$$

see Brezzi and Fortin [7]. Using the notations,

$$e(\mathbf{u}, \mathbf{v}) = 2\mu \int_\Omega \epsilon(\mathbf{u}) : \epsilon(\mathbf{v}) \, dx,$$

$$b(\mathbf{v}, q) = -\int_\Omega \mathrm{div}\mathbf{v} \; q \, dx, \qquad c(p, q) = \int_\Omega pq \, dx,$$

the problem takes the following form:
Find $(\mathbf{u}, p) \in \mathbf{V} \times U$ such that

$$(3) \quad \begin{cases} e(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = < \mathbf{F}, \mathbf{v} > & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u}, q) - \frac{1}{\lambda} c(p, q) = 0 & \forall q \in U. \end{cases}$$

When $\lambda \to \infty$ (or, equivalently, $\nu \to 1/2$), we obtain the limiting problem for incompressible linear elasticity; we then simply drop the appropriate term in (3).

2.3. The generalized Stokes system. In case of homogeneous Dirichlet boundary conditions on the whole boundary $\partial\Omega$, problem (2) is equivalent to the following generalized Stokes problem (see Brezzi and Fortin [7]):

Find $(\mathbf{u}, p) \in \mathbf{V} \times U$ such that

$$
(4) \qquad \left\{ \begin{array}{llllll} s(\mathbf{u},\mathbf{v}) & + & b(\mathbf{v},p) & = & <\mathbf{F},\mathbf{v}> & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u},q) & - & \frac{1}{\lambda+\mu} c(p,q) & = & 0 & \forall q \in U. \end{array} \right.
$$

Here,

$$
s(\mathbf{u},\mathbf{v}) = \mu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \; dx,
$$

and U is now defined by

$$
U = L_0^2(\Omega) = \{ q \in L^2(\Omega) : \int_\Omega q dx = 0 \},
$$

since it can be shown that the pressure will have a zero mean value as a consequence of $\mathbf{u}$ vanishing on the boundary of Ω. The penalty term in (4) can also originate from stabilization techniques or penalty formulations for Stokes problems. The classical Stokes system, describing the velocity $\mathbf{u}$ and pressure p of a fluid of viscosity μ, can be obtained from (4) by letting $\lambda \to \infty$; again we simply drop one of the terms in formula (4). We refer to Girault and Raviart [18] for an introduction to the Stokes and Navier-Stokes equations and their finite element discretization. See also Yang [40] for an alternative formulation of saddle point problems.

3. Spectral element methods. Let Ω_{ref} be the reference cube $(-1,1)^3$, let $Q_n(\overline{\Omega}_{\text{ref}})$ be the set of polynomials on $\overline{\Omega}_{\text{ref}}$ of degree n in each variable, and let $P_n(\overline{\Omega}_{\text{ref}})$ be the set of polynomials on $\overline{\Omega}_{\text{ref}}$ of total degree n. We assume that the domain Ω can be decomposed into N nonoverlapping finite elements Ω_i, each of which is an affine image of the reference cube. Thus, $\Omega_i = \phi_i(\Omega_{\text{ref}})$, where ϕ_i is an affine mapping. The displacement is discretized, component by component, by conforming spectral elements, i.e. by continuous, piecewise polynomials of degree n:

$$
\mathbf{V}^n = \{ \mathbf{v} \in \mathbf{V} : v_k|_{\overline{\Omega}_i} \circ \phi_i \in Q_n(\overline{\Omega}_{\text{ref}}), \; i = 1, \cdots, N, \quad k = 1,2,3 \}.
$$

The pressure space can be discretized by piecewise polynomials of degree $n-2$:

$$
U^n = \{ q \in L_0^2(\Omega) : q|_{\Omega_i} \circ \phi_i \in Q_{n-2}(\Omega_{\text{ref}}), \; i = 1, \cdots, N \}.
$$

We note that the elements of U^n are discontinuous across the boundaries of the elements Ω_i. This choice for U^n gives us the $Q_n - Q_{n-2}$ method, proposed by Maday, Patera, and Rønquist [25] for the Stokes system; see further Subsection 3.3 for a discussion of the stability of this method.

Another choice of the discrete pressure space is given by piecewise polynomials of total degree $n-1$:

$$
\{ q \in U : q|_{\Omega_i} \circ F_i \in P_{n-1}(\Omega_{ref}). \, i = 1, \cdots, N \}.
$$

This choice has been analyzed in Stenberg and Suri [39] and is known as the $Q_n - P_{n-1}$ method. For P_{n-1} a standard tensorial basis does not exist but other bases, common in the p-version finite element literature, can be used. We will not work extensively with this space in this paper.

Other interesting choices for U^n have been studied in Canuto [8] and Canuto and Van Kemenade [9] in connection with stabilization techniques for spectral elements using bubble functions.

3.1. GLL quadrature. Denote by $\{\xi_i, \xi_j, \xi_k\}_{i,j,k=0}^n$ the set of GLL points of the reference cube $[-1,1]^3$, and by σ_i the quadrature weight associated with ξ_i. Let $l_i(x)$ be the Lagrange interpolating polynomial that vanishes at all the GLL nodes except ξ_i where it equals one. The basis functions on the reference cube are then defined by a tensor product as

$$l_i(x)l_j(y)l_k(z), \qquad 0 \le i,j,k \le n.$$

This is a nodal basis, since any element of $Q_n(\Omega_{\text{ref}})$ can be written as

$$u(x,y,z) = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} u(\xi_i,\xi_j,\xi_k)l_i(x)l_j(y)l_k(z).$$

The reference element can be decomposed into its interior, six faces, twelve edges, and eight vertices. The union of its edges and vertices is called the wire basket of the element and is denoted by W_{ref}. Analogously, each basis function can be characterized as being of interior, face, edge, or vertex type:

- interior: $i,j,k \neq 0$ and $\neq n$;
- face: exactly one index is 0 or n;
- edge: exactly two indices are 0 and/or n;
- vertex: all three indices are 0 and/or n.

Each component of the displacement model, and generally any element in V^n, can be written as the sum of its interior, face, edge, and vertex components,

$$u = u_I + u_F + u_E + u_V,$$

where each term is expressed in terms of the corresponding set of basis functions.

For the space U^n, we can similarly use the very convenient basis consisting of tensor-product Lagrangian nodal basis functions associated with just the internal GLL nodes; we note that the degree of the polynomials are now $n-2$. Another basis associated with Gauss-Legendre nodes has been considered in [17] and [24].

We now replace each integral of the continuous models (3) and (4) by using GLL quadrature. On Ω_{ref},

$$(u,v)_{n,\Omega_{\text{ref}}} = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} u(\xi_i,\xi_j,\xi_k)v(\xi_i,\xi_j,\xi_k)\sigma_i\sigma_j\sigma_k,$$

and in general on Ω

$$(u,v)_{n,\Omega} = \sum_{s=1}^{N} \sum_{i,j,k=0}^{n} (u \circ \phi_s)(\xi_i, \xi_j, \xi_k)(v \circ \phi_s)(\xi_i, \xi_j, \xi_k)|J_s|\sigma_i\sigma_j\sigma_k,$$

where $|J_s|$ is the determinant of the Jacobian of ϕ_s. The first inner product is uniformly equivalent to the standard L_2–inner product on $Q_n(\Omega_{\text{ref}})$. Thus, it is shown in Bernardi and Maday [3, 4] that

$$\|u\|^2_{L_2(\Omega_{\text{ref}})} \le (u,u)_{n,\Omega_{\text{ref}}} \le 27\|u\|^2_{L_2(\Omega_{\text{ref}})} \qquad \forall u \in Q_n(\Omega_{\text{ref}}). \tag{5}$$

These bounds imply an analogous uniform equivalence between the $L_2(\Omega)$-norm (and the $H^1(\Omega)$-seminorm) and the corresponding discrete norm (and seminorm) based on GLL quadrature.

3.2. The discrete problems. Applying GLL quadrature to the pure displacement model (1), we obtain the discrete bilinear form

$$a_n(\mathbf{u},\mathbf{v}) = 2\mu(\epsilon(\mathbf{u}) : \epsilon(\mathbf{v}))_{n,\Omega} + \lambda(\text{div}\mathbf{u}, \text{div}\mathbf{v})_{n,\Omega},$$

and the *discrete elasticity system in pure displacement form*:
Find $\mathbf{u} \in \mathbf{V}^n$ such that

$$a_n(\mathbf{u},\mathbf{v}) = <\mathbf{F},\mathbf{v}>_{n,\Omega} \qquad \forall \mathbf{v} \in \mathbf{V}^n. \tag{6}$$

An analysis of the spectral element discretization for the Laplacian and Stokes problems can be found in Bernardi and Maday [3, 4] and in Maday, Patera, and Rønquist [25]. The same techniques can be applied to provide an analysis and error estimates for the linear elasticity problem. The stiffness matrix K associated to the discrete problem (6) is symmetric and positive definite. It is less sparse than the stiffness matrices obtained by low-order finite elements, but still well-structured, and the corresponding matrix-vector multiplication is relatively inexpensive if advantage is taken of its tensor product structure; see, e.g., Bernardi and Maday [3].

For an interior element, $a_n(\cdot,\cdot)$ has a six-dimensional null space $\mathcal{N}$, spanned by the rigid body motions $\mathbf{r}_j$:

$$\mathcal{N} = span\{\mathbf{r}_j,\ j = 1, \cdots, 6\}.$$

On Ω_{ref}, the $\mathbf{r}_j$ are given, component-wise, by three translations

$$\mathbf{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{7}$$

and three rotations

$$\mathbf{r}_4 = \begin{bmatrix} 0 \\ x_3 \\ -x_2 \end{bmatrix}, \quad \mathbf{r}_5 = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}, \quad \mathbf{r}_6 = \begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix}. \tag{8}$$

It is easy to show that both the divergence and the linearized strain tensor of these six functions vanish.

Applying GLL quadrature to the mixed models (3) and (4), we obtain the discrete bilinear forms

$$\begin{aligned} e_n(\mathbf{u},\mathbf{v}) &= 2\mu(\epsilon(\mathbf{u}):\epsilon(\mathbf{v}))_{n,\Omega}, & s_n(\mathbf{u},\mathbf{v}) &= \mu(\nabla\mathbf{u}:\nabla\mathbf{v})_{n,\Omega}, \\ b_n(\mathbf{u},p) &= -(\mathrm{div}\mathbf{u},p)_{n,\Omega}, & c_n(p,q) &= (p,q)_{n,\Omega}. \end{aligned}$$

We note that, since GLL quadrature in each variable is exact for polynomials of degree up to and including $2n-1$ and we are using affine images of the reference cube, the last two bilinear forms are exact, i.e. $b_n(\mathbf{u},p) = b(\mathbf{u},p)$ and $c_n(p,q) = c(p,q), \quad \forall \mathbf{u} \in \mathbf{V}^n, \; p,q \in U^n$.

We can now obtain the *discrete elasticity system in mixed form*: Find $(\mathbf{u},p) \in \mathbf{V}^n \times U^n$ such that

$$(9) \qquad \begin{cases} e_n(\mathbf{u},\mathbf{v}) + b_n(\mathbf{v},p) = <\mathbf{F},\mathbf{v}>_{n,\Omega} & \forall \mathbf{v} \in \mathbf{V}^n \\ b_n(\mathbf{u},q) - \dfrac{1}{\lambda} c_n(p,q) = 0 & \forall q \in U^n \end{cases}$$

In the incompressible case, we remove the $c_n(\cdot,\cdot)$ term, since $1/\lambda = 0$. The *discrete generalized Stokes problem* is an analogous saddle point problem, with $s_n(\cdot,\cdot)$ in place of $e_n(\cdot,\cdot)$ and the penalty parameter equal to $1/(\lambda+\mu)$.

These are all saddle point problems, and they include a penalty term in the elasticity and generalized Stokes case. Using, for simplicity, the same notation for functions and their coefficient vectors, we can write the saddle point problems in matrix form as

$$(10) \qquad K \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & -t^2 C \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix},$$

where A, B, and C are the matrices associated with $s_n(\cdot,\cdot)$ or $e_n(\cdot,\cdot)$, and with $b_n(\cdot,\cdot)$, and $c_n(\cdot,\cdot)$, respectively. The penalty parameter is $t^2 = \dfrac{1}{\lambda}$ for elasticity problems and $t^2 = \dfrac{1}{\lambda+\mu}$ for generalized Stokes problems. The stiffness matrix K is now symmetric and indefinite.

In the following, we will also use $c > 0$ and $C < +\infty$ to denote generic constants in our inequalities; it will be clear from the context if we are referring to generic constants or to the bilinear form $c(\cdot,\cdot)$ and the associated matrix C.

3.3. The inf-sup condition for spectral elements. The convergence of mixed methods depends not only on the approximation properties of the discrete spaces $\mathbf{V}^n$ and U^n, but also on a stability condition known as the inf-sup (or LBB) condition; see, e.g., Brezzi and Fortin [7]. While many important h-version finite elements for Stokes problems satisfy the