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Systems of Quasilinear Equations and Their Applications to Gas Dynamics

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American Mathematical Society · Providence · Rhode Island

СИСТЕМЫ КВАЗИЛИНЕЙНЫХ УРАВНЕНИЙ И ИХ ПРИЛОЖЕНИЯ К ГАЗОВОЙ ДИНАМИКЕ

ИЗДАНИЕ ВТОРОЕ

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И Н. Н. ЯНЕНКО

«НАУКА»
ГЛАВНАЯ РЕДАКЦИЯ
ФИЗИКО-МАТЕМАТИЧЕСКОЙ ЛИТЕРАТУРЫ
МОСКВА 1978

Translated from the Russian by J. R. Schulenberger

1980 *Mathematics Subject Classification.*
Primary 35L60, 35L45, 76N15, 65M10; Secondary 35D99.

ABSTRACT. This book is essentially a new edition, revised and augmented by results of the last decade, of the work of the same title published in 1968 by "Nauka". It is devoted to mathematical questions of gas dynamics.

In Chapter 1 the theory of systems of quasilinear equations, the basic mathematical apparatus of gas dynamics, is presented. Chapter 2 contains a consideration of the main problems of one-dimensional gas dynamics, while Chapter 3 is an account of difference methods. The last, fourth chapter is devoted to the theory of discontinuous solutions of systems of quasilinear equations.

Bibliography: 325 titles.

Library of Congress Cataloging in Publication Data

Rozhdestvenskiĭ, B. L. (Boris Leonidovich)

Systems of quasilinear equations and their applications to gas dynamics.

(Translations of mathematical monographs, ISSN 0065-9282; v. 55)

Translation of: Sistemy kvazilineĭnykh uravneniĭ i ikh prilozheniia k gazovoĭ dinamike.

Bibliography: p.

1. Differential equations, Partial. 2. Gas dynamics. I. Íanenko, N. N. (Nikolaĭ Nikolaevich) II. Title. III. Series.

QA374.R8213 1983 533'.2'01515353 82-24488

ISBN 0-8218-4509-8

PREFACE TO THE SECOND EDITION

Ten years have passed since the appearance of the first edition of our book. In this time many new and interesting results on the questions considered in the book have been obtained, especially on difference methods of solving problems of mathematical physics and gas dynamics.

Some corrections of the text have been inserted in the second edition, and also a considerable number of additions to reflect the progress that has taken place. But as in the first edition, the general plan and style of the exposition affects the choice of material. This applies particularly to the new material in Chapter 3, which is devoted to difference methods. The great number of papers on difference methods, together with the necessarily small size of Chapter 3, have compelled us to omit a number of theoretical questions already covered in sufficient detail in accessible monographs and textbooks.

In preparing the second edition we have again made use of the help of our friends and colleagues, as well as our students. We express our profound gratitude to all of them.

FROM THE PREFACE TO THE FIRST EDITION

In writing the book the authors interacted with various collectives of Soviet mathematicians. Among them we mention those headed by M. V. Keldyš, A. N. Tihonov and A. A. Samarskii, and I. M. Gel'fand. The interaction with friends and professional colleagues inevitably affected our opinions and points of view; a number of results became known to us through conversations with them.

During the course of a number of years each of us gave special courses for students on the theme of this book. As a result of work on the book, several new results were obtained which are published here for the first time.

The present book arose from many years of work during which time we constantly enjoyed the help of many of our friends and professional colleagues and also of many of our students.

We are grateful to A. N. Tihonov, whose advice we constantly followed.

The help of L. V. Ovsjannikov was especially valuable to us; he not only looked through the manuscript of the entire book and made a number of valuable remarks, but also placed at our disposal materials which we used in writing §13 of Chapter 1.

A. A. Samarskii read the manuscript for Chapter 3 and made a number of valuable suggestions.

N. N. Kuznecov helped us a great deal; he read the entire manuscript, made a number of valuable suggestions, and as editor of this book greatly contributed to its improvement.

We express our gratitude to all of them.

INTRODUCTION

To describe the behavior of a continuous medium (a gas, liquid, or solid) theoretical physics uses various models which in most cases lead to nonlinear partial differential and integro-differential equations.

The mechanics of continuous media is the main area of practical application of systems of nonlinear partial differential equations, but it is not the only one. In describing the majority of real physical processes we arrive at nonlinear equations, and only essential additional assumptions regarding the smallness of the amplitudes of waves of the field or the amplitudes of oscillations of the medium, the amplitudes of the deviation for the equilibrium state, etc. lead to linear equations which have been studied more extensively. In Chapter 4 of the book a number of examples are presented of problems of physics, chemistry, and mathematics which are connected with nonlinear equations.

The study of general properties of nonlinear equations and methods of solving them is a rapidly developing area of modern mathematics.

For all the interesting facts and the variety of original and clever methods for investigating and solving nonlinear equations, this area of mathematics does not yet have as solid a theoretical foundation as the theory of linear equations. This is related primarily to the fact that the principle of superposition of solutions is not applicable to nonlinear differential equations, so that the manifold of solutions is not linear.

Systems of quasilinear equations are the simplest hyperbolic systems of nonlinear partial differential equations. Systems of equations with two independent variables have been most studied; these systems describe, in particular, nonstationary one-dimensional and supersonic two-dimensional stationary flows of compressible gases and liquids. However, even for these systems there is presently no sufficiently complete theory; there are no general existence and uniqueness theorems for a solution of the initial value problem.

This is due to the fact that for hyperbolic systems of nonlinear equations the solution of the Cauchy problem involves considerable complication both in the very formulation of the problem and the methods of solving it. Moreover, almost all the basic difficulties that arise here are present already for the case of two independent variables, and it may be expected that solutions of multidimensional equations of gas dynamics locally have essentially the same features as solutions of one-dimensional equations.

The study of hyperbolic systems of nonlinear equations in two independent variables thus constitutes an altogether necessary and so far incomplete stage in the investigation of more general nonlinear equations.

On the basis of these considerations the authors decided to restrict attention mainly to the theory of hyperbolic systems in two independent variables and the study of one-dimensional, nonstationary flows of compressible liquids and gases. Therefore, as a rule, we speak of one of the variables as the time and denote it by the letter t.

Here we shall briefly describe the present state of the question of the solvability of the Cauchy problem for hyperbolic systems of quasilinear equations and the difficulties arising in the attempt to construct a global solution of this problem. The basic method of solving hyperbolic systems of quasilinear equations is the method of characteristics, which is expounded in detail in Chapter 1. The existence, uniqueness, and the continuous dependence on the initial data of a classical solution of the Cauchy problem are proved by means of this method. The results obtained are highly satisfactory in the sense that the classical solution is constructed in the entire domain of the variables t and x where it exists. We observe that the domain of existence of a classical solution is, in general, limited, since solutions of nonlinear equations, in contrast to solutions of linear equations, possess the property of unbounded growth of the magnitude of derivatives which is called the "gradient catastrophe".

The meaning of this property is that even for arbitrarily smooth initial data the first derivatives of a solution become unbounded, in general, in finite time. For some $t_0 > 0$ they become unbounded, and for $t > t_0$ a classical solution of the Cauchy problem does not exist.

From the point of view of gas dynamics this corresponds to the formation of a shock wave (a condensation jump) from the compression wave. Thus, if we wish to determine a solution of the Cauchy problem for any $t \geq 0$, i.e. globally (and this is precisely the problem, for example, in gas dynamics), then we must first find a definition of the solution, since, as already mentioned, a solution of the system of equations in the usual sense—a classical solution—does not exist for $t > t_0$.

In the majority of physical problems and, in particular, in gas dynamics the definition of a generalized solution is dictated by the very formulation of the problem. Thus, for example, in gas dynamics the basic physical laws from which we derive all consequences are the laws of conservation of mass, momentum, and energy. These conservation laws have the character of integral relations, and they are applicable not only to smooth (differentiable) flows. On the other hand, the differential equations of gas dynamics are obtained from these conservation laws under the assumption that the flow is smooth.

We thus define a generalized solution of the equations of gas dynamics as a flow (possibly even with discontinuous parameters) satisfying the basic conservation laws of mass, momentum, and energy. To this we add the thermodynamic requirement that the entropy increase in each thermodynamically closed system. There is a broadly held opinion, which has so far not been contradicted by a single example, that a solution thus defined exists, is unique, and satisfies all reasonable requirements.

The requirement of thermodynamics regarding the increase of entropy is very essential: it indicates the possible direction of the process of rapid variation of the state of the gas. This requirement does not enter in considering classical solutions of the equations of gas dynamics for a gas without viscosity or thermal conductivity, since in smooth flows the entropy of the system is conserved by virtue of the same basic conservation laws.

In gas dynamics another approach to generalized (discontinuous) flows of an ideal gas without viscosity or thermal conductivity is well known. Since a gas without dissipation is an idealization of a gas possessing dissipative processes, it is natural to consider its discontinuous flow as a "limit flow" of a viscous, conducting gas as the coefficients of viscosity and thermal conductivity tend to zero. Here it is assumed that the viscous flows are always described by classical solutions of the differential equations, while the limit as the dissipative coefficients tend to zero exists and is unique in a reasonable sense. This assumption has, in fact, so far not been contradicted by a single example, although precise proofs have presently been obtained only for the very special case of a stationary shock wave.

Here it should be observed that in many cases real gases possess rather small dissipation, so that they can be "approximated" by nondissipative gases. However, the presence of dissipative processes, even small ones, leads to an increase of the entropy of the system. Thus, the requirement of the increase of entropy in the discontinuous flow of an ideal gas is related to the representation of this flow as a "limit" flow of a viscous, thermally conducting gas.

We note that from a mathematical point of view the requirement of the increase of entropy is the condition guaranteeing the uniqueness of a generalized solution and its stability with respect to perturbations.

Although this formulation of the problem of the flow of compressible gases has been known for more than a century (Riemann studied the simplest discontinuous flows), the progress in studying general properties of generalized solutions of the equations of gas dynamics has been relatively minor. Thus, as we have already mentioned, there are still no satisfactory existence and uniqueness theorems.

On the other hand, practical requirements occasioned by the pressing necessity of the practical study of discontinuous flows and also new computing capabilities related to the application of rapid computing technology have led to the situation that, in spite of our inadequate information regarding general properties of discontinuous flows, various numerical algorithms have been created and used which make it possible to satisfactorily compute flows with shock waves. It should be mentioned that in creating these numerical algorithms the majority of the conjectures that we mentioned above were accepted as true.

In view of the fact that a direct and rigorous justification of various assumptions regarding generalized solutions in gas dynamics is a difficult problem, it is natural to hope to verify our views for model equations and systems of equations which to some extent imitate the equations of gas dynamics.

A consequence of this desire was the appearance in the last decade of the so-called *theory of generalized solutions of systems of quasilinear equations* or, more briefly, *the theory of systems of quasilinear equations* (here systems of hyperbolic type are usually meant). This theory poses the problem of introducing in analogy with gas dynamics the concept of a generalized solution for an "arbitrary" system of quasilinear partial differential equations of hyperbolic type, proving its existence, uniqueness, and continuous dependence on the initial data of the problem, and studying the properties of such solutions. At least formally, this theory is more general than one-dimensional gas dynamics and includes the latter as a special case.

It has attracted the attention of many mathematicians, and the number of results obtained through the efforts of Soviet and foreign scientists make it possible to expect further development of the theory.

On the basis of this view of the development of the theory of generalized (discontinuous) solutions of systems of quasilinear equations, the authors limited their attention to the case of only two independent variables and included in the book the following basic questions:

1. Methods of constructing classical solutions of systems of quasilinear equations; proofs of existence and uniqueness theorems and continuous dependence of classical solutions; analytic methods of constructing solutions of systems of nonlinear equations; and conditions for the formation of discontinuities in solutions of arbitrary systems of quasilinear equations. These questions are discussed in Chapter 1. Results obtained for classical solutions of systems of quasilinear equations during recent years are presented there.

2. Classical and generalized solutions of the equations of gas dynamics for

one-dimensional, nonstationary flows. This question is discussed in Chapter 2. The authors considered it expedient to consider in detail certain questions of gas dynamics which are discussed in many textbooks. The foundations of thermodynamics, a derivation of the equations of gas dynamics for various symmetries of a one-dimensional flow, the Hugoniot conditions, generalized properties of flows, the theory of the shock transition, and self-similar and analytic solutions of gas dynamics are presented. The inclusion in the book of these traditional questions of gas dynamics makes it possible to expound from a unified point of view certain mathematical problems which arise in gas dynamics; moreover, the majority of numerical methods in gas dynamics are actually based on this material. The basic problem of the theory of discontinuous solutions of the equations of gas dynanics—the problem of the decay of an arbitrary discontinuity—and also the interaction of shock waves with one another, with travelling waves, and with a contact boundary are considered in detail.

3. Chapter 3 is devoted to difference methods for solving the equations of gas dynamics. In our time these methods have become a basic means of studying problems of gas dynamics, and therefore progress in the study of discontinuous flows is to a considerable extent connected with difference methods.

In this chapter we have had to present the basic concepts of the theory of difference methods. Unfortunately, the majority of assertions of this theory pertain only to the case of linear equations.

The present situation regarding the justification of difference methods applied to the numerical solution of problems of gas dynamics is briefly as follows. Classical solutions (smooth flows) can be computed with almost arbitrary accuracy. The basic method—the numerical method of characteristics—for classical solutions is sufficiently justified. On the other hand, numerical methods applied to compute discontinuous flows have not been rigorously justified, and in the majority of cases certain conjectures regarding the behavior of solutions, the approximation of certain equations by others, etc. are used. Simple equations for which the behavior of the discontinuous solution is well known are most frequently used to verify the various assumptions. It is no accident that in this chapter in most cases each scheme is subject to verification on a simple quasilinear equation with a solution which can be written out explicitly.

This situation regarding the justification of difference methods indicates that progress in this area is to a considerable extent connected with progress in the study of general properties of generalized solutions of systems of quasilinear equations and, in particular, the equations of gas dynamics. On the other hand, difference methods provide experimental material and strongly stimulate the development of the theory of generalized solutions.

4. Chapter 4 is devoted to the theory of generalized solutions of systems of

quasilinear equations of hyperbolic type, and contains the basic results obtained in this area in recent years. A major success here is the construction of a theory of a generalized solution of a single quasilinear equation, which may be considered near completion. For this equation existence, uniqueness, and continuous dependence of the generalized solution of the initial data are proved, and the equivalence of the definitions of a generalized solution from the point of view of the conservation law on the one hand and as a limit of "viscous solutions" on the other is demonstrated.

On the other hand, as in gas dynamics, the study of generalized solutions of systems of equations encounters major difficulties, and very meager results have so far been obtained here. The basic problem, which is currently being subjected to thorough investigation, is the problem of the decay of an arbitrary discontinuity. By means of this simple problem it is possible to study the structure of the generalized solution, and on the basis of this structure it is even possible to construct generalized solutions for the case of a system of two equations.

It should be noted that in recent years more general problems for systems of quasilinear equations have also been studied intensively.

In Chapter 4 the basic results obtained for a single quasilinear equation are presented, the problem of the decay of a discontinuity for an arbitrary hyperbolic system of quasilinear equations is considered, and some results pertaining to more general cases are presented. To conclude this chapter a number of problems are described in various areas of science which are related to the theory of systems of quasilinear equations and, in particular, to discontinuous solutions of such equations.

From what has been said above it should be clear that the mathematical theory of discontinuous solutions of systems of quasilinear equations and, in particular, of the equations of gas dynamics, although it contains many remarkable results and achievements, is far from complete. We hope that our book will afford the reader an idea of the modern methods of solving and studying systems of quasilinear equations and will at the same time spur him to further investigations in this interesting and rapidly developing area of applied mathematics.

The book is subdivided into chapters, sections, and subsections. Formulas are numbered independently in each subsection, and hence references contain the number of the section and subsection in addition to the number of the formula, so that formula (2.7.18) signifies formula (18) in Subsection 7 of Section 2 (i.e., §2.7) of the given chapter. The number of the formula alone is given only in the case where the reference does not go beyond the confines of the given subsecton.

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