Numerical Solutions of Three Classes of Nonlinear Parabolic Integro-Differential Equations

Temur Jangveladze
Zurab Kiguradze
Beny Neta

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Temur Jangveladze
Ilia Vekua Institute of Applied Mathematics of
Ivane Javakhishvili Tbilisi State University
Tbilisi, Georgia
&
Georgian Technical University
Tbilisi, Georgia

Zurab Kiguradze
Ilia Vekua Institute of Applied Mathematics of
Ivane Javakhishvili Tbilisi State University
Tbilisi, Georgia

Beny Neta
Naval Postgraduate School
Department of Applied Mathematics
Monterey, CA, U.S.A.

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PREFACE

This book is concerned with the numerical solutions of some classes of nonlinear integro-differential models. Some properties of the solutions for investigated equations are also given. Three types of nonlinear integro-differential models are considered. Algorithms of finding approximate solutions are constructed and investigated. Results of numerical experiments with graphical illustrations and their analysis are given. The book consists of four Chapters.

In the first Chapter three models (Model I, Model II and Model III, that will be detailed further) to be discussed are introduced and a brief history of integro-differential equations is given.

In the second Chapter, mathematical modeling of a process of penetration of an electromagnetic field into a substance by integro-differential models is described. A short description of the integro-differential equation that is a special model for one-dimensional heat flow in materials with memory is also given in the second Chapter. This model arises in the theory of one-dimensional viscoelasticity as well. This Chapter closes with some concluding remarks of the three investigated models.

The third Chapter is devoted to the numerical solution of the initial-boundary value problems for models stated in the previous Chapter. Semi-discrete schemes and finite-difference approximations, as well as finite elements are discussed. The mathematical substantiation of all these questions for initial-boundary value problems is given.

The questions of the realizations of algorithms investigated in the third Chapter are discussed in the fourth Chapter. Results of the many numerical experiments with graphical illustrations and their analysis are also given in this Chapter.

At the end of the book a list of the quoted literature and indexes are given. The list of references is not intended to be an exhaustive bibliography on the subject, but it is nevertheless detailed enough to enable further independent work.

Each Chapter is concluded with a detailed section, entitled "Comments and bibliographical notes," containing references to the principal results treated, as well as information on important topics related to, but sometimes not included in the body of the text.

The authors believe that the book will be useful to scientists working in the field of nonlinear integro-differential models. In the opinion of the authors, the material presented in the book is helpful for a wide range of readers engaged in mathematical physics, in problems of applied and numerical mathematics, and also MS and PhD students of the appropriate specializations.

Temur Jangveladze,

Zurab Kiguradze,

Beny Neta

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Contents

Abstract

This book is concerned with the numerical solutions of some classes of nonlinear integro-differential models. Some properties of the solutions of the corresponding initial-boundary value problems studied in the monograph equations are given. Three types of nonlinear integro-differential models are considered. Algorithms of finding approximate solutions are constructed and investigated. Results of numerical experiments with tables and graphical illustrations and their analysis are given. The book consists of four chapters. At the end of the book a list of the quoted literature and indexes are given. Each chapter is concluded with a detailed section, entitled "Comments and bibliographical notes," containing references to the principal results treated, as well as information on important topics related to, but sometimes not included in the body of the text.

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Key words: Electromagnetic field penetration, Maxwell's equations, integro-differential models, existence and uniqueness, asymptotic behavior, semi-discrete and finite difference schemes, Galerkin's method, finite element approximation, error estimate, stability and convergence.

Chapter 1

Introduction

Abstract

The description of various kinds of integro-differential equations and a brief history of their origin and applications are given. The importance of investigations of integro-differential models is pointed out as well. Classification of integro-differential equation is given. The main attention is paid on parabolic type integro-differential models. In particular, three types of integro-differential equations are considered. Two of them are based on Maxwell's equations describing electromagnetic field penetration into a substance. The third one is obtained by simulation of heat flow. At the end of the chapter, as at the end of each chapter, the comments and bibliographical notes is given, which consists of description of references concerning to the topic considered.

Key words: Electromagnetic field penetration, Maxwell's system, heat flow equation, integro-differential models.

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In mathematical modeling of applied tasks differential, integral, and integro-differential (I-D, for short) equations appear very often. There are numerous scientific works devoted to the investigation of differential equations. There is a vast literature in the field of integral and integro-differential models as well.

The differential equations are connecting unknown functions, their derivatives, and independent variables. On the other hand, integral equations contain the unknown functions under an integral as well.

The term integro-differential equation in the literature is used in the case when the equation contains unknown function together with its derivatives and when either unknown function, or its derivatives, or both appear under an integral.

Let us recall the general classification of integro-differential equations. If the equation contains derivatives of unknown function of one variable then the integro-differential equation is called ordinary integro-differential equation. The order of an equation is the same as the highest-order derivative of the unknown function in the equation.

The integro-differential equations often encountered in mathematics and physics contain derivatives of various variables; therefore, these equations are called integro-differential equations with partial derivatives or partial integro-differential equations.

In the applications very often there are integro-differential equations with partial derivatives and multiple integrals as well, for example, Boltzmann equation [66] and Kolmogorov-Feller equation [288].

Volterra is one of the founders of the theory of integral and integro-differential equations. His works, especially in the integral and integro-differential equations, are often cited till today. The classical book by Volterra [469] is widely quoted in the literature. In 1884 Volterra [465] began his research in the theory of integral equations devoted to distribution of an electrical charge on a spherical patch. This work led to the equation, which in the modern literature is called the integral equation of the first kind with symmetric kernel.

The work on the theory of elasticity became the beginning research of Volterra leading to the theory of partial integro-differential equations. In 1909 Volterra [466] has studied a particular type of such equations and has shown that this integro-differential equation is equivalent to a system consisting of three linear integral equations and a second order partial differential equations.

The first examples of integro-differential equations with partial derivatives investigated in the beginning of the twentieth century were in Schlesinger's works [417], [418], where the following equation is investigated:

$$\frac{\partial U(x,y)}{\partial x} = \int_a^b f(x,y,s)U(x,s)ds.$$

Numerous works in the beginning of the twentieth century were devoted to research of integro-differential equations of various kinds. The excellent bibliography in this case is given in the classical book by Volterra [469]. In addition, Kerimov [271], the editor of the Russian translation of this book, has updated (up to 1970s) the list of references on integral and integro-differential equations.

Let us describe some classes of mathematical models of second order promoting intensive research on partial integro-differential equations.

When we take into account hereditary phenomena, the questions of physics and mechanics lead to integro-differential equations. A hereditary phenomenon occurs in a system when the phenomenon does not depend only on the actual state of the system but on all the preceding states through which the system has passed; that is to say, it depends on the history of the system and may therefore be called hereditary.

One of the important representatives of an integro-differential equations of elliptic type is the following equation connected with the hereditary phenomenon [467]

$$\Delta U(x,t) + \int_0^t \sum_{i=1}^{3} \frac{\partial^2 U(x,\tau)}{\partial x_i^2} f_i(t,\tau)d\tau = 0,$$

where $x = (x_1, x_2, x_3)$, Δ is a classical three-dimensional Laplace operator

$$\Delta U(x,t) = \frac{\partial^2 U(x,t)}{\partial x_1^2} + \frac{\partial^2 U(x,t)}{\partial x_2^2} + \frac{\partial^2 U(x,t)}{\partial x_3^2},$$

and f_i are known functions of their arguments. Let us note that here and below everywhere instead of x, y, z we use x_1, x_2, x_3 for the designation of Cartesian coordinates.

The mathematical modeling of the vibrations of an elastic chord in the case of linear hereditary process gives a hyperbolic type integro-differential equation [467]

$$\frac{\partial^2 U(x,t)}{\partial t^2} = \frac{\partial^2 U(x,t)}{\partial x^2} + \int_0^t \frac{\partial^2 U(x,\tau)}{\partial x^2}\phi(t,\tau)d\tau, \qquad (1.1)$$

where ϕ is a known function of its arguments.

One of the most important representatives of integro-differential models is the following nonlinear equation describing string vibration obtained by Kirchhoff [286] in 1876

$$\frac{\partial^2 U(x,t)}{\partial t^2} - \left[\lambda + \frac{2}{\pi}\int_0^\pi \left(\frac{\partial U(x,t)}{\partial x}\right)^2 dx\right] \frac{\partial^2 U(x,t)}{\partial x^2} = 0, \quad (1.2)$$

where λ is a known constant. Many authors investigated equation (1.2) and its natural generalizations:

$$\frac{\partial^2 U(x,t)}{\partial t^2} - a\left[\int_0^\pi \left(\frac{\partial U(x,t)}{\partial x}\right)^2 dx\right] \frac{\partial^2 U(x,t)}{\partial x^2} = 0 \qquad (1.3)$$

and

$$\frac{\partial^2 U(x,t)}{\partial t^2} - a\left(\left\|A^{\frac{1}{2}} U(t)\right\|\right) AU(t) = 0, \qquad (1.4)$$

where $a(S) \geq a_0 = const > 0$ is a known function of its argument and A is a self-adjoint positive operator, i.e., $A = A^* > 0$. The norm used in (1.4) is the one defined on the range of the operator A.

In investigating (1.3) and (1.4) type models it is sufficient to mention the following publications: [9], [24], [36], [37], [55], [61], [65], [130], [309], [327], [328], [349], [355], [356], [357], [373], [380], [383], [390], [394], [395] though this list is not complete. Let us note that equations (1.3) and (1.4) are also called Kirchhoff equations. They, along with some similar equations, describe important physical processes, among which are linear and nonlinear dynamics of different dimensional bodies (see, for example, [36], [355], [373] and [471]).

In other questions connected with hereditary phenomena, one finds the integro-differential equations of a parabolic type, which were investigated by Evans [160]. These equations look like

$$\frac{\partial U(x,t)}{\partial t} - \frac{\partial^2 U(x,t)}{\partial x^2} + \int_{t_0}^{t} \frac{\partial^2 U(x,\tau)}{\partial x^2} A(t,\tau)d\tau = 0, \qquad (1.5)$$

where A is a known function of its arguments.

Integro-differential equations of parabolic type arise in the study of various problems in physics, chemistry, technology, economics, etc. One very important problem of applied type is generated by mathematical modeling of processes of electromagnetic field penetration into a substance and is described by the well-known Maxwell's equations [300]. In the works [187] and [188], complex system corresponding to nonlinear partial-differential equations was reduced to integro-differential form. If the coefficient of thermal heat capacity and electroconductivity of the substance are highly dependent on temperature, then the Maxwell's system can be rewritten in the following form (see [187] and [188]):

$$\frac{\partial W(x,t)}{\partial t} + \nabla \times [a(S(x,t))\nabla \times W(x,t)] = 0, \qquad (1.6)$$

$$\nabla \cdot W(x,t) = 0,$$

where

$$S(x,t) = \int_0^t |\nabla \times W(x,\tau)|^2 \, d\tau. \qquad (1.7)$$

In system (1.6), $\nabla \times W$ and $\nabla \cdot W$ are the usual vector operators with respect to the variables $x = (x_1, x_2, x_3)$. Even one-dimensional scalar version of this model is very complicated and its investigation has been possible yet only for special cases. The one-dimensional scalar case of the model (1.6), (1.7) has the following form

$$\frac{\partial U(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[a\left(\int_0^t \left|\frac{\partial U(x,\tau)}{\partial x}\right|^2 d\tau\right)\frac{\partial U(x,t)}{\partial x}\right], \qquad (1.8)$$

where $a(S) \geq a_0 = const > 0$ is again a known function of its argument. Investigation of (1.6), (1.7), and (1.8) type models