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E. B. Vinberg (Ed.)

Geometry II
Spaces of Constant Curvature

几 何 II
常曲率空间

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

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Foreword to the English Edition

The authors have introduced some improvements into the English edition. The literature list has been enlarged (mainly that to Part 2) and a number of short additions have been included in the text referring, in particular, to recent works which appeared after the publication of the Russian edition. We have also made every effort to correct misprints and occasional inaccuracies in the Russian version.

The process of translating the book into English has revealed that the use of geometric terminology in the English language literature on the subject is rather haphazard (which, incidentally, is also the case in Russian). In each case, after consulting many relevant English language publications, we and the translator have made our choice. However, in the footnotes we provide alternative terms also used in the literature.

The authors are extremely grateful to Dr. Minachin who performed the laborious work of translating the book with a sense of responsibility and interest.

E. B. Vinberg

Contents

I. Geometry of Spaces of Constant Curvature

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A.S. Solodovnikov

Translated from the Russian
by V. Minachin

Contents

Preface

Spaces of constant curvature, i.e. Euclidean space, the sphere, and Lobachevskij space, occupy a special place in geometry. They are most accessible to our geometric intuition, making it possible to develop elementary geometry in a way very similar to that used to create the geometry we learned at school. However, since its basic notions can be interpreted in different ways, this geometry can be applied to objects other than the conventional physical space, *the original source of our geometric intuition.*

Euclidean geometry has for a long time been deeply rooted in the human mind. The same is true of spherical geometry, since a sphere can naturally be embedded into a Euclidean space. Lobachevskij geometry, which in the first fifty years after its discovery had been regarded only as a logically feasible by-product appearing in the investigation of the foundations of geometry, has even now, despite the fact that it has found its use in numerous applications, preserved a kind of exotic and even romantic element. This may probably be explained by the permanent cultural and historical impact which the proof of the independence of the Fifth Postulate had on human thought.

Nowadays modern research trends call for much more businesslike use of Lobachevskij geometry. The traditional way of introducing Lobachevskij geometry, based on a kind of Euclid-Hilbert axiomatics, is ill suited for this purpose because it does not enable one to introduce the necessary analytical tools from the very beginning. On the other hand, introducing Lobachevskij geometry starting with some specific model also leads to inconveniences since different problems require different models. The most reasonable approach should, in our view, start with an axiomatic definition, but it should be based on a well-advanced system of notions and make it possible either to refer to any *model or do without any model* at all.

Their name itself provides the description of the property by which spaces of constant curvature are singled out among Riemannian manifolds. However, another characteristic property is more important and natural for them — the property of maximum mobility. This is the property on which our exposition is based.

The reader should realize that our use of the term "space of constant curvature" does not quite coincide with the conventional one. Usually one understands it as describing any Riemannian manifold of constant curvature. Under our definition (see Chap. 1, Sect. 1) any space of constant curvature turns out to be one of the three spaces listed at the beginning of the Preface.

Although Euclidean space is, of course, included in our exposition as a special case, we have no intention of introducing the reader to Euclidean geometry. On the contrary, we make free use of its basic facts and theorems. We also assume that the reader is familiar with the basics of linear algebra and affine geometry, the notion of a smooth manifold and Lie group, and the elements of Riemannian geometry.

For the history of non-Euclidean geometry and the development of its ideas the reader is referred to relevant chapters in the books of Klein [1928], Kagan [1949, 1956], Coxeter [1957], and Efimov [1978].

Chapter 1
Basic Structures

§ 1. Definition of Spaces of Constant Curvature

This chapter provides the definition of spaces of constant curvature and of their basic structures, and describes their place among homogeneous spaces on the one hand and Riemannian manifolds on the other. If the reader's main aim is just to study Lobachevskij geometry, no great damage will be done if he skips Theorems 1.2 and 1.3 and the proof of Theorem 2.1.

1.1. Lie Groups of Transformations. We assume that the reader is familiar with the notions of a (real) smooth manifold and of a (real) Lie group. The word "smooth" (manifold, function, map etc.) always means that the corresponding structure is C^∞. All smooth manifolds are assumed to have a countable base of open subsets. By $T_x(X)$ we denote the tangent space to a manifold X at a point x, and by $d_x g$ the differential of the map g at a point x. If no indication of the point is necessary the subscript is omitted.

We now recall some basic definitions of Lie group theory. (For more details see, e.g. Vinberg and Onishchik [1988].)

A group G of transformations[1] of a smooth manifold X endowed with a Lie group structure is said to be a *Lie group of transformations* of the manifold X if the map

$$G \times X \to X, \qquad (g, x) \mapsto gx,$$

is smooth, which means that the (local) coordinates of the point gx are smooth functions of the coordinates of the element g and the point x. Then the stabilizer

$$G_x = \{g \in G : gx = x\}$$

of any point $x \in X$ is a (closed) Lie subgroup of the group G. Its linear representation $g \mapsto d_x g$ in the space $T_x(X)$ is called the *isotropy representation* and the linear group $d_x G_x$ is called the *isotropy group* at the point x.

The stabilizers of equivalent points x and $y = gx \quad (g \in G)$ are conjugate in G, i.e.

$$G_y = gG_x g^{-1}.$$

The corresponding isotropy groups are related in the following way:

$$d_x G_y = (d_x g)(d_x G_x)(d_x g)^{-1}.$$

In other words, if tangent spaces $T_x(X)$ and $T_y(Y)$ are identified by the isomorphism $d_x g$, then the group $d_x G_x$ coincides with the group $d_y G_y$.

[1] By a group of transformations we understand an effective group of transformations, i.e. we assume that different transformations correspond to different elements of the group.

If G is a transitive Lie group of transformations of a manifold X, then for each point $x \in X$ the map

$$G/G_x \to X, \qquad gG_x \mapsto gx$$

is a diffeomorphism commuting with the action of the group G. (The group G acts on the manifold G/G_x of left cosets by left shifts.) In this case the manifold X together with the action of G on it can be reconstructed from the pair (G, G_x).

Definition 1.1. A smooth manifold X together with a given transitive Lie group G of its transformations is said to be a *homogeneous space.*

We denote a homogeneous space by (X, G), or simply X.

A homogeneous space (X, G) is said to be connected or simply-connected[2] if the manifold X has this property.

1.2. Group of Motions of a Riemannian Manifold. A Riemannian metric is said to be defined on a smooth manifold X if a Euclidean metric is defined in each tangent space $T_x(X)$, and if the coefficients of this metric are smooth functions in the coordinates of x. A diffeomorphism g of a Riemannian manifold X is called a *motion* (or an *isometry*) if for each point $x \in X$ the linear map

$$d_x g : T_x(X) \to T_{gx}(X)$$

is an isometry. The set of all motions is evidently a group.

Each motion g takes a geodesic into a geodesic, and therefore commutes with the exponential map, i.e.

$$g(\exp \xi) = \exp dg(\xi)$$

for all $\xi \in T(X)$. Hence each motion g of a connected manifold X is uniquely defined by the image gx of some point $x \in X$ and the differential $d_x g$ at that point. This enables us to introduce coordinates into the group of motions, turning it into a Lie group. To be more precise, the following theorem holds.

Theorem 1.2 (Kobayashi and Nomizu [1981]). *The group of motions of a Riemannian manifold X is uniquely endowed with a differentiable structure, which turns it into a Lie group of transformations of the manifold X.*

If the group of motions of a Riemannian manifold X is transitive, then X is complete. Indeed, in this case there exists $\varepsilon > 0$, which does not depend on x, such that for any point $x \in X$ and for any direction at that point there exists a geodesic segment of length ε issuing from x in that direction. This implies that each geodesic can be continued indefinitely in any direction.

A Riemannian manifold X is said to have *constant curvature* c if at each point its sectional curvature along any plane section equals c.

[2] We assume that any simply-connected space is, by definition, connected.

Simply-connected complete Riemannian manifolds of constant curvature admit a convenient characterization in terms of the group of motions.

Theorem 1.3 (Wolf [1972]). *A simply-connected complete Riemannian manifold is of constant curvature if and only if for any pair of points $x, y \in X$ and for any isometry $\varphi : T_x(X) \to T_y(X)$ there exists a (unique) motion g such that $gx = y$ and $d_x g = \varphi$.*

The first part of the statement follows immediately from the fact that motions preserve curvature and that any given two-dimensional subspace of the space $T_x(X)$ can, by an appropriate isometry, be taken into any given two-dimensional subspace of the space $T_y(X)$. For the proof of the converse statement see Chap. 8, Sect. 1.3.

1.3. Invariant Riemannian Metrics on Homogeneous Spaces. Let (X, G) be a homogeneous space. A Riemannian metric on X is said to be *invariant* (with respect to G) if all transformations in G are motions with respect to that metric. An invariant Riemannian metric can be reconstructed from the Euclidean metric it defines on any tangent space $T_x(X)$. This Euclidean metric is invariant under the isotropy group $d_x G_x$. Conversely, if a Euclidean metric is defined in the space $T_x(X)$ and is invariant under the isotropy group, then it can be moved around by the action of the group G thus yielding an invariant Riemannian metric on X. Thus, an invariant Riemannian metric on X exists if and only if there is a Euclidean metric in the tangent space invariant under the isotropy group.

We now consider the question of when such a metric is unique.

A linear group H acting in a vector space V is said to be irreducible if there is no non-trivial subspace $U \subset V$ invariant under H.

Lemma 1.4. *Let H be a linear group acting in a real vector space V. If H is irreducible, then up to a (positive) scalar multiple there is at most one Euclidean metric in the space V invariant under H.*

Proof. Consider any invariant Euclidean metric (if such a metric exists) turning V into a Euclidean space. Then each invariant Euclidean metric q on V is of the form $q(x) = (Ax, x)$, where A is a positive definite symmetric operator commuting with all operators in H. Let c be any eigenvalue of A. The corresponding eigenspace is invariant under H, and consequently coincides with V. This implies that $A = cE$, i.e. $q(x) = c(x, x)$. □

The Lemma implies that if the isotropy group of a homogeneous space is irreducible, then there exists, up to a (positive) scalar multiple, at most one invariant Riemannian metric.

If a homogeneous space X is connected and admits an invariant Riemannian metric, then the isotropy representation is faithful at each point $x \in X$, since each element of the stabilizer of x, being a motion, is uniquely defined by its differential at that point.