STOCHASTIC MODELLING AND APPLIED PROBABILITY

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Philip E. Protter

Stochastic Integration and Differential Equations

Second Edition, Version 2.1

随机积分和微分方程

第2版

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Preface to the Second Edition

It has been thirteen years since the first edition was published, with its subtitle "a new approach." While the book has had some success, there are still almost no other books that use the same approach. (See however the recent book by K. Bichteler [17].) There are nevertheless of course other extant books, many of them quite good, although the majority still are devoted primarily to the case of continuous sample paths, and others treat stochastic integration as one of many topics. Examples of alternative texts which have appeared since the first edition of this book are: [34], [46], [90], [115], [202], [196], [224], [235], and [248]. While the subject has not changed much, there have been new developments, and subjects we thought unimportant in 1990 and did not include, we now think important enough either to include or to expand in this book.

The most obvious changes in this edition are that we have added exercises at the end of each chapter, and we have also added Chap. VI which introduces the expansion of filtrations. However we have also completely rewritten Chap. III. In the first edition we followed an elementary approach which was P. A. Meyer's original approach before the methods of Doléans-Dade. In order to remain friends with Freddy Delbaen, and also because we now agree with him, we have instead used the modern approach of predictability rather than naturality. However we benefited from the new proof of the Doob-Meyer Theorem due to R. Bass, which ultimately uses only Doob's quadratic martingale inequality, and in passing reveals the role played by totally inaccessible stopping times. The treatment of Girsanov's theorem now includes the case where the two probability measures are not necessarily equivalent, and we include the Kazamaki-Novikov theorems. We have also added a section on compensators, with examples. In Chap. IV we have expanded our treatment of martingale representation to include the Jacod-Yor Theorem, and this has allowed us to use the Emery-Azéma martingales as a class of examples of martingales with the martingale representation property. Also, largely because of the Delbaen-Schachermayer theory of the fundamental theorems of mathematical finance, we have included the topic of sigma martingales. In Chap. V we added a section which includes some useful results about the solutions of stochastic differential equations, inspired by the review of the first edition by E. Pardoux [207]. We have also made small changes throughout the book; for instance we have included specific examples of Lévy processes and their corresponding Lévy measures, in Sect. 4 of Chap. I.

The exercises are gathered at the end of the chapters, in no particular order. Some of the (presumed) harder problems we have designated with a star (*), and occasionally we have used two stars (**). While of course many of the problems are of our own creation, a significant number are theorems or lemmas taken from research papers, or taken from other books. We do not attempt to ascribe credit, other than listing the sources in the bibliography, primarily because they have been gathered over the past decade and often we don't remember from where they came. We have tried systematically to refrain from relegating a needed lemma as an exercise; thus in that sense the exercises are independent from the text, and (we hope) serve primarily to illustrate the concepts and possible applications of the theorems.[1]

Last, we have the pleasant task of thanking the numerous people who helped with this book, either by suggesting improvements, finding typos and mistakes, alerting me to references, or by reading chapters and making comments. We wish to thank patient students both at Purdue University and Cornell University who have been subjected to preliminary versions over the years, and the following individuals: C. Beneš, R. Cont, F. Diener, M. Diener, R. Durrett, T. Fujiwara, K. Giesecke, L. Goldberg, R. Haboush, J. Jacod, H. Kraft, K. Lee, J. Ma, J. Mitro, J. Rodriguez, K. Schürger, D. Sezer, J. A. Trujillo Ferreras, R. Williams, M. Yor, and Yong Zeng. Th. Jeulin, K. Shimbo, and Yan Zeng gave extraordinary help, and my editor C. Byrne gives advice and has patience that is impressive. Over the last decade I have learned much from many discussions with Darrell Duffie, Jean Jacod, Tom Kurtz, and Denis Talay, and this no doubt is reflected in this new edition. Finally, I wish to give a special thanks to M. Kozdron who hastened the appearance of this book through his superb help with LaTeX, as well as his own advice on all aspects of the book.

This postscript concerns the Corrected Second Edition. Since the appearance of the second edition, Marc Yor has read the book with care and made many suggestions which have been incorporated in this corrected edition. Many are subtle, but without doubt the reader will benefit greatly from them, and we wish to thank him for this gift. I am also grateful for help received from others, including K. Asrat, K. Shimbo, and Y. Zeng.

Ithaca, NY
February 2005

Philip Protter

[1] Solutions of some of the exercises are posted on the author's web page, URL http://www.orie.cornell.edu/~protter/books.html (July, 2004).

Preface to the First Edition

The idea of this book began with an invitation to give a course at the Third Chilean Winter School in Probability and Statistics, at Santiago de Chile, in July, 1984. Faced with the problem of teaching stochastic integration in only a few weeks, I realized that the work of C. Dellacherie [44] provided an outline for just such a pedagogic approach. I developed this into a series of lectures (Protter [217]), using the work of K. Bichteler [16], E. Lenglart [158] and P. Protter [218], as well as that of Dellacherie. I then taught from these lecture notes, expanding and improving them, in courses at Purdue University, the University of Wisconsin at Madison, and the University of Rouen in France. I take this opportunity to thank these institutions and Professor Rolando Rebolledo for my initial invitation to Chile.

This book assumes the reader has some knowledge of the theory of stochastic processes, including elementary martingale theory. While we have recalled the few necessary martingale theorems in Chap. I, we have not provided proofs, as there are already many excellent treatments of martingale theory readily available (e.g., Breiman [25], Dellacherie-Meyer [47, 48], or Ethier-Kurtz [74]). There are several other texts on stochastic integration, all of which adopt to some extent the usual approach and thus require the general theory. The books of Elliott [66], Kopp [138], Métivier [174], Rogers-Williams [226] and to a much lesser extent Letta [162] are examples. The books of McKean [169], Chung-Williams [34], and Karatzas-Shreve [129] avoid the general theory by limiting their scope to Brownian motion (McKean) and to continuous semimartingales.

Our hope is that this book will allow a rapid introduction to some of the deepest theorems of the subject, without first having to be burdened with the beautiful but highly technical "general theory of processes."

Many people have aided in the writing of this book, either through discussions or by reading one of the versions of the manuscript. I would like to thank J. Azema, M. Barlow, A. Bose, M. Brown, C. Constantini, C. Dellacherie, D. Duffie, M. Emery, N. Falkner, E. Goggin, D. Gottlieb, A. Gut, S. He, J. Jacod, T. Kurtz, J. de Sam Lazaro, R. Léandre, E. Lenglart, G. Letta,

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Svante Janson read the entire manuscript in several versions, giving me support, encouragement, and wonderful suggestions, all of which improved the book. He also found, and helped to correct, several errors. I am extremely grateful to him, especially for his enthusiasm and generosity.

The National Science Foundation provided partial support throughout the writing of this book.

I wish to thank Judy Snider for her cheerful and excellent typing of several versions of this book.

Philip Protter

Stochastic Mechanics
Random Media
Signal Processing
and Image Synthesis
Mathematical Economics and Finance
Stochastic Optimization
Stochastic Control
Stochastic Models in Life Sciences

Stochastic Modelling and Applied Probability

(Formerly:
Applications of Mathematics)

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Contents

Introduction

In this book we present a new approach to the theory of modern stochastic integration. The novelty is that we define a semimartingale as a stochastic process which is a "good integrator" on an elementary class of processes, rather than as a process that can be written as the sum of a local martingale and an adapted process with paths of finite variation on compacts: This approach has the advantage over the customary approach of not requiring a close analysis of the structure of martingales as a prerequisite. This is a significant advantage because such an analysis of martingales itself requires a highly technical body of knowledge known as "the general theory of processes." Our approach has a further advantage of giving traditionally difficult and non-intuitive theorems (such as Stricker's Theorem) transparently simple proofs. We have tried to capitalize on the natural advantage of our approach by systematically choosing the simplest, least technical proofs and presentations. As an example we have used K. M. Rao's proofs of the Doob-Meyer decomposition theorems in Chap. III, rather than the more abstract but less intuitive Doléans-Dade measure approach.

In Chap. I we present preliminaries, including the Poisson process, Brownian motion, and Lévy processes. Naturally our treatment presents those properties of these processes that are germane to stochastic integration.

In Chap. II we define a semimartingale as a good integrator and establish many of its properties and give examples. By restricting the class of integrands to adapted processes having left continuous paths with right limits, we are able to give an intuitive Riemann-type definition of the stochastic integral as the limit of sums. This is sufficient to prove many theorems (and treat many applications) including a change of variables formula ("Itô's formula").

Chapter III is devoted to developing a minimal amount of "general theory" in order to prove the Bichteler-Dellacherie Theorem, which shows that our "good integrator" definition of a semimartingale is equivalent to the usual one as a process X having a decomposition $X = M + A$, into the sum of a local martingale M and an adapted process A having paths of finite variation on compacts. Nevertheless most of the theorems covered en route (Doob-

Meyer, Meyer-Girsanov) are themselves key results in the theory. The core of the whole treatment is the Doob-Meyer decomposition theorem. We have followed the relatively recent proof due to R. Bass, which is especially simple for the case where the martingale jumps only at totally inaccessible stopping times, and in all cases uses no mathematical tool deeper than Doob's quadratic martingale inequality. This allows us to avoid the detailed treatment of natural processes which was ubiquitous in the first edition, although we still use natural processes from time to time, as they do simplify some proofs.

Using the results of Chap. *III* we extend the stochastic integral by continuity to predictable integrands in Chap. IV, thus making the stochastic integral a Lebesgue-type integral. We use predictable integrands to develop a theory of martingale representation. The theory we develop is an L^2 theory, but we also prove that the dual of the martingale space $\mathcal{H}^1$ is BMO and then prove the Jacod-Yor Theorem on martingale representation, which in turn allows us to present a class of examples having both jumps and martingale representation. We also use predictable integrands to give a presentation of semimartingale local times.

Chapter V serves as an introduction to the enormous subject of stochastic differential equations. We present theorems on the existence and uniqueness of solutions as well as stability results. Fisk-Stratonovich equations are presented, as well as the Markov nature of the solutions when the differentials have Markov-type properties. The last part of the chapter is an introduction to the theory of flows, followed by moment estimates on the solutions, and other minor but useful results. Throughout Chap. V we have tried to achieve a balance between maximum generality and the simplicity of the proofs.

Chapter VI provides an introduction to the theory of the expansion of filtrations (known as "grossissements de filtrations" in the French literature). We present first a theory of initial expansions, which includes Jacod's Theorem. Jacod's Theorem gives a sufficient condition for semimartingales to remain semimartingales in the expanded filtration. We next present the more difficult theory of progressive expansion, which involves expanding filtrations to turn a random time into a stopping time, then analyzing what happens to the semimartingales of the first filtration when considered in the expanded filtration. Last, we give an application of these ideas to time reversal.

I

Preliminaries

1 Basic Definitions and Notation

We assume as given a complete probability space $(\Omega, \mathcal{F}, P)$. In addition we are given a *filtration* $(\mathcal{F}_t)_{0 \le t \le \infty}$. By a filtration we mean a family of σ-algebras $(\mathcal{F}_t)_{0 \le t \le \infty}$ that is increasing, i.e., $\mathcal{F}_s \subset \mathcal{F}_t$ if $s \le t$. For convenience, we will usually write $\mathbb{F}$ for the filtration $(\mathcal{F}_t)_{0 \le t \le \infty}$.

Definition. A filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is said to satisfy the **usual hypotheses** if

(i) $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$;
(ii) $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$, all t, $0 \le t < \infty$; that is, the filtration $\mathbb{F}$ is *right continuous*.

We always assume that the usual hypotheses hold.

Definition. A random variable $T : \Omega \to [0, \infty]$ is a **stopping time** if the event $\{T \le t\} \in \mathcal{F}_t$, every t, $0 \le t \le \infty$.

One important consequence of the right continuity of the filtration is the following theorem.

Theorem 1. *The event* $\{T < t\} \in \mathcal{F}_t$, $0 \le t \le \infty$, *if and only if* T *is a stopping time.*

Proof. Since $\{T \le t\} = \bigcap_{t+\varepsilon>u>t}\{T < u\}$, any $\varepsilon > 0$, we have $\{T \le t\} \in \bigcap_{u>t} \mathcal{F}_u = \mathcal{F}_t$, so T is a stopping time. For the converse, $\{T < t\} = \bigcup_{t>\varepsilon>0}\{T \le t - \varepsilon\}$, and $\{T \le t - \varepsilon\} \in \mathcal{F}_{t-\varepsilon}$, hence also in $\mathcal{F}_t$. □

A **stochastic process** X on $(\Omega, \mathcal{F}, P)$ is a collection of $\mathbb{R}$-valued or $\mathbb{R}^d$-valued random variables $(X_t)_{0 \le t < \infty}$. The process X is said to be **adapted** if $X_t \in \mathcal{F}_t$ (that is, is $\mathcal{F}_t$ measurable) for each t. We must take care to be precise about the concept of equality of two stochastic processes.

Definition. Two stochastic processes X and Y are **modifications** if $X_t = Y_t$ a.s., each t. Two processes X and Y are **indistinguishable** if a.s., for all t, $X_t = Y_t$.

If X and Y are *modifications* there exists a null set, N_t, such that if $\omega \notin N_t$, then $X_t(\omega) = Y_t(\omega)$. The null set N_t depends on t. Since the interval $[0,\infty)$ is uncountable the set $N = \bigcup_{0\leq t<\infty} N_t$ could have any probability between 0 and 1, and it could even be non-measurable. If X and Y are *indistinguishable*, however, then there exists one null set N such that if $\omega \notin N$, then $X_t(\omega) = Y_t(\omega)$, for all t. In other words, the functions $t \mapsto X_t(\omega)$ and $t \mapsto Y_t(\omega)$ are the same for all $\omega \notin N$, where $P(N) = 0$. The set N is in $\mathcal{F}_t$, all t, since $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$. The functions $t \mapsto X_t(\omega)$ mapping $[0,\infty)$ into $\mathbb{R}$ are called the **sample paths** of the stochastic process X.

Definition. A stochastic process X is said to be **càdlàg** if it a.s. has sample paths which are right continuous, with left limits. Similarly, a stochastic process X is said to be **càglàd** if it a.s. has sample paths which are left continuous, with right limits. (The nonsensical words *càdlàg* and *càglàd* are acronyms from the French for *continu à droite, limité à gauche* and *continu à gauche, limité à droite*, respectively.)

Theorem 2. *Let X and Y be two stochastic processes, with X a modification of Y. If X and Y have right continuous paths a.s., then X and Y are indistinguishable.*

Proof. Let A be the null set where the paths of X are not right continuous, and let B be the analogous set for Y. Let $N_t = \{\omega : X_t(\omega) \neq Y_t(\omega)\}$, and let $N = \bigcup_{t\in\mathbb{Q}} N_t$, where $\mathbb{Q}$ denotes the rationals in $[0,\infty)$. Then $P(N) = 0$. Let $M = A \cup B \cup N$, and $P(M) = 0$. We have $X_t(\omega) = Y_t(\omega)$ for all $t \in \mathbb{Q}$, $\omega \notin M$. If t is not rational, let t_n decrease to t through $\mathbb{Q}$. For $\omega \notin M$, $X_{t_n}(\omega) = Y_{t_n}(\omega)$, each n, and $X_t(\omega) = \lim_{n\to\infty} X_{t_n}(\omega) = \lim_{n\to\infty} Y_{t_n}(\omega) = Y_t(\omega)$. Since $P(M) = 0$, X and Y are indistinguishable. □

Corollary. Let X and Y be two stochastic processes which are càdlàg. If X is a modification of Y, then X and Y are indistinguishable.

Càdlàg processes provide natural examples of stopping times.

Definition. Let X be a stochastic process and let Λ be a Borel set in $\mathbb{R}$. Define

$$T(\omega) = \inf\{t > 0 : X_t \in \Lambda\}.$$

Then T is called the **hitting time** of Λ for X.

Theorem 3. *Let X be an adapted càdlàg stochastic process, and let Λ be an open set. Then the hitting time of Λ is a stopping time.*

Proof. By Theorem 1 it suffices to show that $\{T < t\} \in \mathcal{F}_t$, $0 \leq t < \infty$. But

$$\{T < t\} = \bigcup_{s\in\mathbb{Q}\cap[0,t)} \{X_s \in \Lambda\},$$

since Λ is open and X has right continuous paths. Since $\{X_s \in \Lambda\} = X_s^{-1}(\Lambda) \in \mathcal{F}_s$, the result follows. □