CLASSICS IN MATHEMATICS

David Mumford

Algebraic Geometry I Complex Projective Varieties

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Complex Projective Varieties

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Introduction

Let me begin with a little history. In the 20th century, algebraic geometry has gone through at least 3 distinct phases. In the period 1900–1930, largely under the leadership of the 3 Italians, Castelnuovo, Enriques and Severi, the subject grew immensely. In particular, what the late 19th century had done for curves, this period did for surfaces: a deep and systematic theory of surfaces was created. Moreover, the links between the "synthetic" or purely "algebro-geometric" techniques for studying surfaces, and the topological and analytic techniques were thoroughly explored. However the very diversity of tools available and the richness of the intuitively appealing geometric picture that was built up, led this school into short-cutting the fine details of all proofs and ignoring at times the time-consuming analysis of special cases (e.g., possibly degenerate configurations in a construction). This is the traditional difficulty of geometry, from High School Euclidean geometry on up. In the period 1930–1960, under the leadership of Zariski, Weil, and (towards the end) Grothendieck, an immense program was launched to introduce systematically the tools of commutative algebra into algebraic geometry and to find a common language in which to talk, for instance, of projective varieties over characteristic p fields as well as over the complex numbers. In fact, the goal, which really goes back to Kronecker, was to create a "geometry" incorporating at least formally arithmetic as well as projective geometry. Several ways of achieving this were proposed, but after a somewhat chaotic period in which communication was difficult, it seems fair to say that Grothendieck's "schemes" have become generally accepted as providing the most satisfactory foundations. In the present period 1960 on, algebraic geometry is growing rapidly in many directions at once: to a deeper understanding of geometry in dimensions higher than 2, especially their singularities, and the theory of cycles on them; to uncovering the astonishing connections between the topology of varieties and their Diophantine properties (their rational points over finite fields and number fields); and to the theory of moduli, i.e., the parameters describing continuous families of varieties.

To acquire a good understanding of modern algebraic geometry, the insights of each of these periods have to be studied. In particular, it is necessary to know something both of classical projective geometry, of curves and surfaces in complex projective space and the "synthetic" tools for manipulating them (such as linear systems)—this amounts to what people call "geometric intuition"—and to know something of the analogies between arithmetic and geometry, of "Spec" and of "specialization mod p". Moreover, it is necessary to know both how algebraic and differential topology and complex analytic tools (such as Hodge theory) apply to complex varieties; and to know how commutative algebra can be used. It is not clear where to start! I have given introductory lectures on algebraic geometry on at least 5 separate occasions. I have taken a different tack each time, and there are several other approaches I would like to try in the future. This book grew out of notes that went through several nearly total revisions as a consequence. In the end, I found it impractical to teach classical geometry and schemes at the same time. Therefore, the present volume, which is the first of several, introduces only complex projective varieties. But, as a consequence, we can study these effectively with topological and analytic techniques without extensive preliminary work on "foundations". My goal is precisely to convey some of the classical geometric ideas and to get "off the ground": in fact, to get to the 27 lines on the cubic—surely one of the gems hidden in the rag-bag of projective geometry. The next volume will deal with schemes, including cohomology of coherent sheaves on them and applications, e.g., to π_1 of curve. The pedagogical difficulty here is that the definition itself of schemes is hard to swallow; and technically a massive amount of commutative algebra is needed to get schemes off the ground. My hope is that a previous acquaintance with complex projective varieties provides motivation and intuition for schemes.

A detailed list of prerequisites for this book follows this introduction. I hope it is almost entirely a subset of the list of "standard results" which are generally common property to all pure mathematicians. My goal has been not only to write a text for graduate students but to open the subject to specialists in other areas. Algebraic geometry is a subject that thrives on exchanging ideas and not on isolation and should be more universally understood! In particular, I have tried to write a book which you can browse in as well as read linearly. For some general guide to other literature, let me mention:

i) for the 1900–1930 phase, Severi [1]*, Semple-Roth [1], and Zariski [1]

ii) for the foundational phase, Grothendieck's tome [EGA] is standard for schemes but very hard to read. Another classic is Samuel [1].

iii) among recent books, there is Serre [1], and Šafarevič [2]. An introductory book by R. Hartshorne is expected. An excellent survey of recent research is the publication AMS [1].

Tenants Harbor,
August, 1975

David Mumford

Note for Second Printing

The author would like to thank Dr. Ronald Infante and Mr. Francis McGuinness for compiling long lists of misprints, inconsistent notations, and other oversights in the first printing of this book. Hopefully, their corrections will make this printing easier to read.

DM

*All references are given in full in the bibliography.

Prerequisites

Algebraic geometry is not a "primary" mathematical subject, i.e., one which one builds directly from a small and elegant set of axioms or definitions. This makes it very hard to write an introductory book accessible to the 1st year graduate student. In general, this book is aimed at 2nd year students or anyone with at least some basic familiarity with topology, differential and analytic geometry, and commutative algebra. I want to list here all the concepts and theorems which will be assumed known at one point or another in this book (except for the results used only in the more difficult Appendix to Chapter 6).

I. *Topology:*

a) Besides standard elementary point set topology, the concept of a covering space is frequently used, e.g., in §§3A, 4B, 7D and 8D.

b) The classification of compact, orientable surfaces by the "number of handles" is used in §7B.

c) The homology groups (singular homology) come in twice: in §5C, in dealing with minimal submanifolds in $\mathbb{P}^n$, and in §7B, in calculating via Euler characteristic the number of handles of a smooth algebraic curve regarded as a topological space.

II. *Differential Geometry:*

Basic advanced calculus knowledge of differential forms and Stoke's theorem is used in 2 ways:

a) We assume DeRham's theorem that the periods of integrals give a perfect duality between real homology and closed forms mod exact forms in §5C.

b) We deduce from Stoke's theorem the basic properties of the residue at a pole of an analytic 1-form on a 1-dimensional complex manifold in §7C.

III. *Analytic Geometry:*

a) In §1B, we recall very quickly the definition of complex manifold, and we assume known the implicit function theorem for analytic functions. For our purposes, an analytic function is by definition a function given locally by a convergent power series of several complex variables.

b) In §4A, and §4B, we use repeatedly the fundamental local fact of analytic geometry: the Weierstrass Preparation theorem. Since this may be used to deduce the implicit function theorem, this includes the assumptions just above.

c) Once in §4A we use the fact that a complex-valued function f with $\mathrm{Re}(f)$ and $\mathrm{Im}\, f$ differentiable, satisfying the Cauchy-Riemann equations, is analytic.

A good reference for all this material is Gunning-Rossi [1].

IV. *Commutative Algebra:*

I wish I were able to cut down these prerequisites considerably, but I have been drawn into assuming the following:

a) from field theory, the concept of transcendence degree and its connection with derivations,

b) from general ring theory, the concepts of localizing a ring and a module, the concept of local ring, graded ring and graded module and the concept of the completion of a local ring. Integral dependence and integral closure are *not* used, except in a minor digression in §6C giving a second proof of one theorem.

c) Resultants are used in §2C and §4A to give elementary constructive proofs of several theorems. The basic facts about the resultant are summarized in §2C.

d) The decomposition theorem of ideals in noetherian rings is used in §1A and §4B in its really elementary form: if $\mathfrak{A} = \sqrt{\mathfrak{A}}$, then $\mathfrak{A} = \mathfrak{P}_1 \cap \ldots \cap \mathfrak{P}_s$, $\mathfrak{P}_i$ prime.

e) Several explicit facts about the formal power series ring $\mathbb{C}[[X_1,\ldots,X_n]]$ are used in Chapter I: that it is a UFD, and the formal implicit function theorem. This last is a very elementary special case of the formal Weierstrass Preparation theorem and could really be "left to the reader" as an exercise.

f) Finally, we have assumed in §§1A, 1C and 7B Krull's theorem: that if R is a noetherian local ring, $M \subset R$ its maximal ideal, $I \subset R$ any ideal, then

$$\bigcap_{n=1}^{\infty} (I + M^n) = I$$

This can however be easily deduced from the ideal decomposition theorem (see $Z-S$, vol. I, p. 217). If

$$\hat{R} \underset{\text{def}}{=} \varprojlim_{n} R/M^n$$

is the completion of R, then the theorem is equivalent to saying:

$$R \cap I \cdot \hat{R} = I.$$

It would have been nice to avoid using this somewhat less generally known result: but I don't know any straightforward way of proving Theorem (1.16) without it.

My standard reference for commutative algebra is $Z-S$, (both volumes), which was, in fact, written to be background for a book on algebraic geometry.

Table of Contents

Chapter 1. Affine Varieties

§1A. Their Definition, Tangent Space, Dimension, Smooth and Singular Points

The beginning of the whole subject is the following definition:

(1.1) **Definition.** *A closed algebraic subset X of $\mathbb{C}^n$ is the set of zeroes of a finite set of polynomials $f_1,\ldots,f_m$, i.e., the set of all $x=(x_1,\ldots,x_n)$ such that $f_i(x)=0$, $1\leqq i\leqq m$. We denote X by $V(f_1,\ldots,f_n)$.*

If $\mathfrak{A}=(f_1,\ldots,f_m)$ is the ideal in $\mathbb{C}[X_1,\ldots,X_n]$ generated by $f_1,\ldots,f_m$, then the set of zeroes of the f_i is also the set of zeroes of every $g\in\mathfrak{A}$, so we will denote X also by $V(\mathfrak{A})$. Note that there would be no point in definition (1.1) in using infinite sets of f_i's, since the ideal they generate would also be generated by a finite subset of them according to Hilbert's basis theorem; hence the set of zeroes of all the f_i's would equal the set of zeroes of this finite subset. We get immediately the following properties

a) $\mathfrak{A}_1\subseteq\mathfrak{A}_2\Longrightarrow V(\mathfrak{A}_1)\supseteq V(\mathfrak{A}_2)$,

b) $V(\mathfrak{A}_1)\cup V(\mathfrak{A}_2)=V(\mathfrak{A}_1\cap\mathfrak{A}_2)=V(\mathfrak{A}_1\cdot\mathfrak{A}_2)$

c) $V(\sum_{\alpha\in J}\mathfrak{A}_\alpha)=\bigcap_{\alpha\in J}V(\mathfrak{A}_\alpha)$,

d) If $\mathfrak{M}_x=$ the maximal ideal $(X_1-x_1,\ldots,X_n-x_n)$, then $V(\mathfrak{M}_x)=\{x\}$.

As a consequence of (b) and (c), the subsets $V(\mathfrak{A})$ of $\mathbb{C}^n$ satisfy the axioms for the closed sets of a topology. The topology of $\mathbb{C}^n$ with these as closed sets will be called the *Zariski topology* as opposed to the usual topology.

e) If $\sqrt{\mathfrak{A}}=\{f\in\mathbb{C}[X]\,|\,f^m\in\mathfrak{A}$, some $m\geqq 1\}$ is the so-called radical of $\mathfrak{A}$, then $V(\sqrt{\mathfrak{A}})=V(\mathfrak{A})$.

Now according to a standard result in the theory of noetherian rings (cf. $Z-S$, vol. 1, p. 209) an ideal which equals its own radical is a finite irredundant intersection of prime ideals in a unique way:

(*) If $\mathfrak{A}=\sqrt{\mathfrak{A}}$, then $\mathfrak{A}=\mathfrak{P}_1\cap\ldots\cap\mathfrak{P}_k$, where $\mathfrak{P}_i\not\supseteq\mathfrak{P}_j$ if $i\neq j$. This proves:

f) For any ideal $\mathfrak{A}$, let $\sqrt{\mathfrak{A}}=\mathfrak{P}_1\cap\ldots\cap\mathfrak{P}_k$, then

$$V(\mathfrak{A})=V(\mathfrak{P}_1)\cup\ldots\cup V(\mathfrak{P}_k).$$

(1.2) **Definition.** *A closed algebraic set $V(\mathfrak{P})$, where $\mathfrak{P}$ is a prime ideal, is called an affine variety.*

Examples: a) if $f\in\mathbb{C}[X_1,\ldots,X_n]$ is an irreducible polynomial, then the principal ideal (f) is prime, hence $V(f)$ is a variety. Such varieties are called *hypersurfaces.*

b) let $g_2, \dots, g_n \in \mathbb{C}[X_1]$ be polynomials. Consider the set

$$X = \{(a, g_2(a), \dots, g_n(a)) \mid a \in \mathbb{C}\}.$$

Then $X = V(X_2 - g_2(X_1), \dots, X_n - g_n(X_1))$. The ideal $\mathfrak{A} = (X_2 - g_2, \dots, X_n - g_n)$ is prime since it is the kernel of the homomorphism

$$\begin{aligned} \mathbb{C}[X_1, \dots, X_n] &\longrightarrow \mathbb{C}[X_1] \\ X_1 &\longrightarrow X_1 \\ X_i &\longrightarrow g_i(X_1), \quad i \geqq 2. \end{aligned}$$

Therefore X is a variety. It is a simple type of *rational space curve.*

c) Let $l_1, \dots, l_k$ be independent linear forms in $X_1, \dots, X_n$. Let $a_1, \dots, a_k \in \mathbb{C}$. Then $X = V(l_1 - a_1, \dots, l_k - a_k)$ is a variety, called a *linear subspace* of $\mathbb{C}^n$ of dimension $n - k$.

A basic idea in the classical theory is the following:

(1.3) **Definition.** *Let $k \subset \mathbb{C}$ be a subfield, and let $\mathfrak{P}$ be a prime ideal. A k-generic point $x \in V(\mathfrak{P})$ is a point such that every polynomial $f(X_1, \dots, X_n)$ with coefficients in k that vanishes at x is in the ideal $\mathfrak{P}$, hence vanishes on all of X.*

Example: In example (b) above if the coefficients of the g_i are in $\mathbb{Q}$, the point $(\pi, g_2(\pi), \dots, g_n(\pi))$ is a $\mathbb{Q}$-generic point of this rational curve.

(1.4) **Proposition.** *If $\mathbb{C}$ has infinite transcendence degree over k, then every variety $V(\mathfrak{P})$ has a k-generic point.*

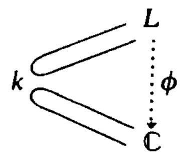
Proof. Let $f_1, \dots, f_m$ be generators of $\mathfrak{P}$. We may enlarge k if we wish by adjoining the coefficients of all the f_i without destroying the hypothesis. Let

$$\mathfrak{P}_0 = \mathfrak{P} \cap k[X_1, \dots, X_n]$$

and let

$$L = \text{quotient field of } k[X_1, \dots, X_n]/\mathfrak{P}_0.$$

Then L is an extension field of k of finite transcendence degree. But any such field is isomorphic to a subfield of $\mathbb{C}$: i.e., $\exists$ a monomorphism ϕ



If $\overline{X}_i$ = image of X_i in L and $a_i = \phi(\overline{X}_i)$, I claim $a = (a_1, \dots, a_n)$ is a k-generic point. In fact, $f_i \in \mathfrak{P}_0$, $1 \leqq i \leqq k$, hence $f_i(\overline{X}_1, \dots, \overline{X}_n) = 0$ in L. Therefore $f_i(a_1, \dots, a_n) = 0$ in $\mathbb{C}$ and a is indeed a point of X. But if $f \in k[X_1, \dots, X_n]$ and $f \notin \mathfrak{P}$, then $f \notin \mathfrak{P}_0$, hence $f(\overline{X}_1, \dots, \overline{X}_n) \neq 0$ in L. Therefore $f(a_1, \dots, a_n) = \phi(f(\overline{X}_1, \dots \overline{X}_n)) \neq 0$ in $\mathbb{C}$.

QED

For any subset $S \subset \mathbb{C}^n$, let $I(S)$ be the ideal of polynomials $f \in \mathbb{C}[X_1, \dots, X_n]$

that vanish at all points of S. Then an immediate corollary of the existence of generic point is:

(1.5) **Hilbert's Nullstellensatz.** *If $\mathfrak{P}$ is a prime ideal, then $\mathfrak{P}$ is precisely the ideal of polynomials $f \in \mathbb{C}[X_1,\ldots,X_n]$ that vanish identically on $V(\mathfrak{P})$, i.e., $\mathfrak{P} = I(V(\mathfrak{P}))$. More generally, if $\mathfrak{A}$ is any ideal, then $\sqrt{\mathfrak{A}} = I(V(\mathfrak{A}))$.*

Proof. Given any $f \in \mathbb{C}[X]$, let k be a finitely generated field over $\mathbb{Q}$ containing the coefficients of f and let $a \in V(\mathfrak{P})$ be a k-generic point. If $f \notin \mathfrak{P}$, then $f(a) \neq 0$ hence f does not vanish identically on $V(\mathfrak{P})$; the 2nd assertion reduces to the 1st by means of (f) on p. 2.

(1.6) **Corollary.** *There is an order-reversing bijection between the set of ideals $\mathfrak{A}$ such that $\mathfrak{A} = \sqrt{\mathfrak{A}}$ and the closed algebraic subsets $X \subset \mathbb{C}^n$ set up by $\mathfrak{A} \longrightarrow V(\mathfrak{A})$ and $X \longrightarrow I(X) =$* (ideals of functions zero on X). *In this bijection, varieties correspond to prime ideals and are precisely the closed algebraic sets which are irreducible (i.e., not the union of two smaller closed algebraic sets).*

(1.7) **Corollary.** *If $X = V(\mathfrak{P})$ is a variety, the ring $\mathbb{C}[X_1,\ldots,X_n]/\mathfrak{P}$ is canonically isomorphic to the ring of functions $X \longrightarrow \mathbb{C}$ which are restrictions of polynomials. This ring is called the affine coordinate ring of X and will be denoted R_X.*

The Nullstellensatz usually found in the literature (e.g., Z-S [1], vol. II, p. 164) applies to varieties over any algebraically closed groundfield k and is much harder to prove than (1.5).

Our main goal in this section is to give a first idea of the structure of affine varieties. Since the simplest type of varieties are the linear ones, we can try to approximate a general variety by a linear one:

(1.8) **Definition.** *Let $X = V(\mathfrak{P})$ be a variety and let $a = (a_1,\ldots,a_n) \in X$. The Zariski tangent space to X at a is the linear subspace of $\mathbb{C}^n$ defined by*

$$\sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(a)\cdot(X_i - a_i) = 0, \qquad \text{all } f \in \mathfrak{P}.$$

We denote this space by $T_{X,a}$ (or $T_{a,X}$).

Note that for each $k \in \mathbb{Z}$, $\{a \in X \mid \dim T_{X,a} \geq k\}$ is a Zariski closed subset of X. In fact, if $f_1,\ldots,f_l$ are generators of $\mathfrak{P}$:

$$\dim T_{X,a} = n - rk\left(\frac{\partial f_i}{\partial X_j}(a)\right)_{\substack{1\leq i\leq l\\ 1\leq j\leq n}}$$

hence

$$\{a \in X \mid \dim T_{X,a} \geq k\} = V\begin{pmatrix}\mathfrak{P} + \text{ideal of } (n-k+1)\times(n-k+1)- \\ \text{minors of } (\partial f_i/\partial X_j)\end{pmatrix}$$

More succinctly, we can say that $\dim T_{X,a}$ is an upper semicontinuous function of a in the Zariski topology. (1.8) treats the tangent space *externally*, i.e., as a subspace of $\mathbb{C}^n$. But if we regard $T_{X,a}$ as an abstract vector space with origin a, we can define it *intrinsically* by derivations on the affine coordinate ring R_X of X.

If $a \in X$, then a derivation $D: R_X \longrightarrow \mathbb{C}$ *centered at a* means a $\mathbb{C}$-linear map such that

i) $D(fg) = f(a)\cdot D(g) + g(a)\cdot D(f)$,

ii) $D(\alpha) = 0$, all $\alpha \in \mathbb{C}$.

Clearly a derivation $D: R_X \longrightarrow \mathbb{C}$ at a is the same thing as a derivation D': $\mathbb{C}[X_1,\ldots,X_n] \longrightarrow \mathbb{C}$ at a such that $D'(f) = 0$, all $f \in \mathfrak{P}$. But a D' is determined by its values $\lambda_i = D(X_i)$ and conversely, given any $\lambda_1,\ldots,\lambda_n$ there is a D' with these values on the X_i. Since

$$D'f = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(a)\cdot DX_i, \qquad \text{all} \quad f \in \mathbb{C}[X],$$

the derivations D' which kill $\mathfrak{P}$ correspond to the $\lambda_1,\ldots,\lambda_n$ such that

$$\sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(a)\cdot \lambda_i = 0, \qquad \text{all} \quad f \in \mathfrak{P}.$$

This proves that

$$(1.9) \qquad T_{X,a}\begin{pmatrix}\text{as vector space}\\ \text{with origin } a\end{pmatrix} \cong \begin{Bmatrix}\text{vector space of derivations}\\ D: R_X \longrightarrow \mathbb{C} \text{ centered at } a\end{Bmatrix}$$

We can make the definition more local too. First introduce:

(1.10) **Definition.** *If $X = V(\mathfrak{P}) \subset \mathbb{C}^n$ is an affine variety and $a \in X$, then define in any of 3 ways(!):*

$\mathcal{O}_{a,X}$ = *ring of rational functions* $\dfrac{f(X_1,\ldots,X_n)}{g(X_1,\ldots,X_n)}$ *where* $g(a) \neq 0$, *modulo those with* $f \in \mathfrak{P}$
= *localization of R_X with respect to the multiplicative set of $g, g(a) \neq 0$*
= *ring of germs of functions $U \longrightarrow \mathbb{C}$, U a Zariski neighborhood of a in X, defined by rational functions f/g, where $g(a) \neq 0$.*

$\mathcal{O}_{a,X}$ is a local ring, since the set of functions f/g which are zero at a forms a maximal ideal and every f/g which is not zero at a is invertible, i.e., $g/f \in \mathcal{O}_{a,X}$. $\mathcal{O}_{a,X}$ is called the local ring of $a \in X$.

The point is that every derivation $D: R_X \longrightarrow \mathbb{C}$ extends uniquely to the ring $\mathcal{O}_{a,X}$ by the rule $D(f/g) = (g(a)Df - f(a)Dg)/g(a)^2$, hence

$$T_{X,a} \cong \begin{Bmatrix}\text{vector space of derivations}\\ D: \mathcal{O}_{a,X} \to \mathbb{C} \text{ centered at } a\end{Bmatrix}.$$

Note that every function $f \in \mathcal{O}_{a,X}$ defines a linear map $df: T_{X,a} \longrightarrow \mathbb{C}$, its *differential* by the rule $df(D) = D(f)$. If $T_{X,a}$ is considered *externally* as a subspace of $\mathbb{C}^n$, then df is nothing but the linear term $\sum \dfrac{\partial f}{\partial X_i}(a)\cdot(X_i - a_i)$ in the Taylor expansion of f at a.

R_X and all the local rings $\mathcal{O}_{x,X}$ have the same quotient field, which we will denote $\mathbb{C}(X)$ and call the *function field* of X. An important fact is that the local rings $\mathcal{O}_{x,X}$ determine the affine ring R_X. In fact:

(1.11) **Proposition.** *Taking intersections in* $\mathbb{C}(X)$, *we have:*

$$R_X = \bigcap_{x \in X} \mathcal{O}_{x,X}.$$

Proof. The inclusion "$\subset$" is clear. Now say $f \in \mathcal{O}_{x,X}$ for all $x \in X$. Consider the ideal in $\mathbb{C}[X_1, \ldots, X_n]$ defined by:

$$\mathfrak{A} = \{g \in \mathbb{C}[X_1, \ldots, X_n] \mid \text{if } \bar{g} = (g \bmod \mathfrak{P}) \in R_X, \quad \bar{g} \cdot f \in R_X\}.$$

Since $f \in \mathcal{O}_{x,X}$ we can write $f = h_x/g_x$, where $h_x, g_x \in \mathbb{C}[X_1, \ldots, X_n], g_x(x) \neq 0$. Thus $g_x \in \mathfrak{A}$, hence $x \notin V(\mathfrak{A})$ for all $x \in X$. Moreover, $\mathfrak{A} \supset \mathfrak{P}$, so $V(\mathfrak{A}) \subset V(\mathfrak{P}) = X$. Therefore $V(\mathfrak{A}) = \phi$. But then by the Nullstellensatz $1 \in I(V(\mathfrak{A})) = \sqrt{\mathfrak{A}}$, hence $1 \in \mathfrak{A}$. By definition, this means $f \in R_X$. QED

We next use the derivations of $\mathbb{C}(X)$ to prove:

(1.12) **Proposition.** $\exists$ *a non-empty Zariski open subset* $U \subset X$ *such that:*

$$\text{tr.d.}_{\mathbb{C}}\mathbb{C}(X) = \dim T_{X,a}, \textit{for all } a \in U.$$

Proof. It is well known that the transcendence degree of any separably generated field extension K/L is equal to the dimension of the K-vector space of derivations $D: K \longrightarrow K$ that kill L. In our case,

$$\begin{aligned}
\text{tr.d.}_{\mathbb{C}}\mathbb{C}(X) &= \dim_{\mathbb{C}(X)}(\text{derivations } D: \mathbb{C}(X) \longrightarrow \mathbb{C}(X) \text{ that kill } \mathbb{C}) \\
&= \dim_{\mathbb{C}(X)}(\text{derivations } D: R_X \longrightarrow \mathbb{C}(X) \text{ that kill } \mathbb{C}) \\
&= \dim_{\mathbb{C}(X)}\begin{pmatrix}\text{derivations } D: \mathbb{C}[X_1, \ldots, X_n] \longrightarrow \mathbb{C}(X) \\ \text{that kill } \mathbb{C} \text{ and } \mathfrak{P}\end{pmatrix} \\
&= \dim_{\mathbb{C}(X)}\begin{pmatrix}n\text{-tuples } \lambda_1, \ldots, \lambda_n \in \mathbb{C}(X) \text{ such that} \\ \sum \frac{\partial f}{\partial X_i} \cdot \lambda_i = 0 \text{ in } \mathbb{C}(X), \text{ for all } f \in \mathfrak{P}\end{pmatrix} \\
&= n - rk\begin{pmatrix}\text{the image in } \mathbb{C}(X) \text{ of matrix} \\ (\partial f_i/\partial X_j)_{\substack{1 \leq j \leq l \\ 1 \leq i \leq n}} \text{ where } \mathfrak{P} = (f_1, \ldots, f_l)\end{pmatrix}
\end{aligned}$$

Therefore it suffices to show

$$\left[rk \text{ in } \mathbb{C}(X) \text{ of } \left(\frac{\partial f_i}{\partial X_j}\right)_{\substack{1 \leq i \leq l \\ 1 \leq j \leq n}}\right] = \left[rk \frac{\partial f_i}{\partial X_j}(a)_{\substack{1 \leq i \leq l \\ 1 \leq j \leq n}}\right],$$

for all a in a Zariski open subset of X. But if r is the rank in $\mathbb{C}(X)$, then there is an invertible $l \times l$ matrix A over $\mathbb{C}(X)$ and an invertible $n \times n$ matrix B over $\mathbb{C}(X)$ such that:

$$A.\left(\frac{\partial f_i}{\partial X_j} \bmod \mathfrak{P}\right).B = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

Write $A = A_0/\alpha, B = B_0/\beta$ where A_0, B_0 are matrices over R_X and $\alpha, \beta \in R_X$. If U is the Zariski open set of X where $\det A_0 \cdot \det B_0 \cdot \alpha \cdot \beta \neq 0$, then for all $a \in U$,