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Symmetry and Integration Methods for Differential Equations

微分方程用的对称和积分方法

George W. Bluman
Stephen C.Anco

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Symmetry and Integration Methods for Differential Equations

With 18 Illustrations

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Preface

This book is a significant update of the first four chapters of *Symmetries and Differential Equations* (1989; reprinted with corrections, 1996), by George W. Bluman and Sukeyuki Kumei. Since 1989 there have been considerable developments in symmetry methods (group methods) for differential equations as evidenced by the number of research papers, books, and new symbolic manipulation software devoted to the subject. This is, no doubt, due to the inherent applicability of the methods to nonlinear differential equations. Symmetry methods for differential equations, originally developed by Sophus Lie in the latter half of the nineteenth century, are highly algorithmic and hence amenable to symbolic computation. These methods systematically unify and extend well-known ad hoc techniques to construct explicit solutions for differential equations, especially for nonlinear differential equations. Often ingenious tricks for solving particular differential equations arise transparently from the symmetry point of view, and thus it remains somewhat surprising that symmetry methods are not more widely known. Nowadays it is essential to learn the methods presented in this book to understand existing symbolic manipulation software for obtaining analytical results for differential equations. For ordinary differential equations (ODEs), these include reduction of order through group invariance or integrating factors. For partial differential equations (PDEs), these include the construction of special solutions such as similarity solutions or nonclassical solutions, finding conservation laws, equivalence mappings, and linearizations.

A large portion of this book discusses work that has appeared since the above-mentioned book, especially connected with finding first integrals for higher-order ODEs and using higher-order symmetries to reduce the order of an ODE. Also novel is a comparison of various complementary symmetry and integration methods for an ODE.

The present book includes a comprehensive treatment of dimensional analysis. There is a full discussion of aspects of Lie groups of point transformations (point symmetries), contact symmetries, and higher-order symmetries that are essential for finding solutions of differential equations. No knowledge of group theory is assumed. Emphasis is placed on explicit algorithms to discover symmetries and integrating factors admitted by a given differential equation and to construct solutions and first integrals resulting from such symmetries and integrating factors.

This book should be particularly suitable for applied mathematicians, engineers, and scientists interested in how to find systematically explicit solutions of differential equations. Almost all examples are taken from physical and engineering problems including those concerned with heat conduction, wave propagation, and fluid flow.

Chapter 1 includes a thorough treatment of dimensional analysis. The well-known Buckingham Pi-theorem is presented in a manner that introduces the reader concretely to the notion of invariance. This is shown to naturally lead to generalizations through invariance of boundary value problems under scalings of variables. This prepares the reader to consider the still more general invariance of differential equations under Lie groups of transformations in the third and fourth chapters. Basically, the first chapter gives the reader an intuitive grasp of some of the subject matter of the book in an elementary setting.

Chapter 2 develops the basic concepts of Lie groups of transformations and Lie algebras that are necessary in the following two chapters. By considering a Lie group of point transformations through its infinitesimal generator from the point of view of mapping functions into functions with their independent variables held fixed, we show how one is able to consider naturally other local transformations such as contact transformations and higher-order transformations. Moreover, this allows us to prepare the foundation for consideration of integrating factors for differential equations.

Chapter 3 is concerned with ODEs. A reduction algorithm is presented that reduces an nth-order ODE, admitting a solvable r-parameter Lie group of point transformations (point symmetries), to an $(n - r)$th-order differential equation and r quadratures. We show how to find admitted point, contact, and higher-order symmetries. We also show how to extend the reduction algorithm to incorporate such symmetries. It is shown how to find admitted first integrals through corresponding integrating factors and to obtain reductions of order using first integrals. We show how this simplifies and significantly extends the classical Noether's Theorem for finding conservation laws (first integrals) to any ODE (not just one admitting a variational principle). In particular, we show how to calculate integrating factors by various algorithmic procedures analogous to those for calculating symmetries in characteristic form where only the dependent variable undergoes a transformation. We also compare the distinct methods of reducing order through admitted local symmetries and through admitted integrating factors. We show how to use invariance under point symmetries to solve boundary value problems. We derive an algorithm to construct special solutions (invariant solutions) resulting from admitted symmetries. By studying their topological nature, we show that invariant solutions include separatrices and singular envelope solutions.

Chapter 4 is concerned with PDEs. It is shown how to find admitted point symmetries and how to construct related invariant solutions. There is a full discussion of the applicability to boundary value problems with numerous examples involving scalar PDEs and systems of PDEs.

Chapters 2 to 4 can be read independently of the first chapter. Moreover, a reader interested in PDEs can skip the third chapter.

Every topic is illustrated by examples. All sections have many exercises. It is essential to do the exercises to obtain a working knowledge of the material. The Discussion section at the end of each chapter puts its contents into perspective by summarizing major results, by referring to related works, and by introducing related material.

Within each section and subsection of a given chapter, we number separately, and consecutively, definitions, theorems, lemmas, and corollaries. For example, Definition 2.3.3-1 refers to the first definition and Theorem 2.3.3-1 to the first theorem in Section 2.3.3. Exercises appear at the conclusion of each section; Exercise 2.4-2 refers to the second problem of Exercises 2.4.

We thank Benny Bluman for the illustrations and Cecile Gauthier for typing several drafts of Sections 3.5 to 3.8.

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Contents

Introduction

In the latter part of the nineteenth century, Sophus Lie introduced the notion of continuous groups, now known as Lie groups, in order to unify and extend various specialized methods for solving ordinary differential equations (ODEs). Lie was inspired by the lectures of Sylow given at Christiania (present-day Oslo) on Galois theory and Abel's related works. [In 1881 Sylow and Lie collaborated in a careful editing of Abel's complete works.] Lie showed that the order of an ODE could be reduced by one, constructively, if it is invariant under a one-parameter Lie group of point transformations.

Lie's work systematically related a miscellany of topics in ODEs including: integrating factor, separable equation, homogeneous equation, reduction of order, the methods of undetermined coefficients and variation of parameters for linear equations, solution of the Euler equation, and the use of the Laplace transform. Lie (1881) also indicated that for linear partial differential equations (PDEs), invariance under a Lie group leads directly to superpositions of solutions in terms of transforms.

A *symmetry* of a system of differential equations is a transformation that maps any solution to another solution of the system. In Lie's framework such transformations are groups that depend on continuous parameters and consist of either point transformations (*point symmetries*), acting on the system's space of independent and dependent variables, or, more generally, contact transformations (*contact symmetries*), acting on the space of independent and dependent variables as well as on all first derivatives of the dependent variables. Elementary examples of Lie groups include translations, rotations, and scalings. An autonomous system of first-order ODEs, i.e., a stationary flow, essentially defines a one-parameter Lie group of point transformations. Lie showed that for a given differential equation (linear or nonlinear), the admitted continuous group of point transformations, acting on the space of its independent and dependent variables, can be determined by an explicit computational algorithm (*Lie's algorithm*).

In this book, the applications of continuous groups to differential equations make no use of the global aspects of Lie groups. These applications use connected local Lie groups of transformations. Lie's fundamental theorems show that such groups are completely characterized by their *infinitesimal generators*. In turn, these form a *Lie algebra* determined by structure constants.

Lie groups, and hence their infinitesimal generators, can be naturally *extended* or *"prolonged"* to act on the space of independent variables, dependent variables, and derivatives of the dependent variables up to any finite order. As a consequence, the seemingly intractable nonlinear conditions for group invariance of a given system of differential equations reduce to linear homogeneous equations determining the infinitesimal generators of the group. Since these *determining equations* form an overdetermined system of linear homogeneous PDEs, one can usually determine the infinitesimal generators in explicit form. For a given system of differential equations, the setting up of the determining equations is entirely routine. Symbolic manipulation programs exist to set up the determining equations and in some cases explicitly solve them [Schwarz (1985, 1988); Kersten (1987); Head (1992); Champagne, Hereman, and Winternitz (1991); Wolf and Brand (1992); Hereman (1996); Reid (1990, 1991); Mansfield (1996); Mansfield and Clarkson (1997); Wolf (2002a)].

One can generalize Lie's work to find and use *higher-order symmetries* admitted by differential equations. The possibility of the existence of higher-order symmetries appears to have been first considered by Noether (1918). Such symmetries are characterized by infinitesimal generators that act only on dependent variables, with coefficients of the generators depending on independent variables, dependent variables and their derivatives to some finite order. Here, unlike the case for point symmetries or contact symmetries, any extension of the corresponding global transformation is not closed on any finite-dimensional space of independent variables, dependent variables and their derivatives to some finite order. In particular, globally, such transformations act on the infinite-dimensional space of independent variables, dependent variables, and their derivatives to all orders. Nonetheless, a natural extension of Lie's algorithm can be used to find such transformations for a given differential equation.

For a first-order ODE, Lie showed that invariance of the ODE under a point symmetry is equivalent to the existence of a first integral for the ODE. In this situation a *first integral* yields a conserved quantity that is constant for each solution of the ODE. Local existence theory for an nth-order ODE shows that there always exist n functionally independent first integrals of the ODE, which are quadratures relating the independent variable, dependent variable and its derivatives to order $n-1$. Correspondingly, an nth order ODE admits n essential conserved quantities. Moreover, it is a long-known result that any first integral arises from an *integrating factor*, given by a function of the independent variable, dependent variable and its derivatives to some order, which multiplies the ODE to transform it into an exact (total derivative) form.

For a higher-order ODE, a correspondence between first integrals and invariance under point symmetries holds only when the ODE has a variational principle (*Lagrangian*). In particular, Noether's work showed that invariance of such an ODE under a point symmetry, a contact symmetry, or a higher-order symmetry is equivalent to the existence of a first integral for the ODE if the symmetry leaves invariant the variational principle of the ODE (*variational symmetry*). Here it is essential to view a symmetry in its *characteristic form* where the coefficient of its infinitesimal generator acts only on the dependent variable (and its derivatives) in the ODE. The determining equation for symmetries is then given by the linearization (Frèchet derivative) of the ODE holding for *all* solutions of the ODE. The condition for a symmetry to be a variational symmetry is expressed by augmenting the linearization of the ODE through extra determining equations. Integrating factors are solutions of the resulting augmented system of determining equations.

For an ODE with no variational principle, we show that integrating factors are related to *adjoint-symmetries* defined as solutions of the adjoint equation of the linearization (Frèchet derivative) of the ODE, holding for all solutions of the ODE. In particular, there are necessary and sufficient extra determining equations for an adjoint-symmetry to be an integrating factor. This generalizes the equivalence between first integrals and variational symmetries in the case of an ODE with a variational principle, to an equivalence between first integrals and adjoint-symmetries that satisfy extra *adjoint invariance conditions* in the case of an ODE with no variational principle.

As a consequence, adjoint-symmetries play a central role in the study of first integrals of ODEs. Most important, an obvious extension of the calculational algorithm for solving the symmetry-determining equation can be used to solve the determining equation for adjoint-symmetries and the augmented system of determining equations for integrating factors.

Integrating factors provide another method for constructively reducing the order of an ODE through finding a first integral. This reduction of order method is complementary to, and independent of, Lie's reduction method for second- and higher-order ODEs. In particular, the integrating factor method is just as algorithmic and no more computationally complex than Lie's algorithm. Moreover, with the integrating factor approach one obtains a reduction of order in terms of the given variables in the original ODE, unlike reduction through point symmetries where the reduced ODE involves derived independent and dependent variables (and usually remains of the same order as the given ODE if expressed in the original variables).

If a system of PDEs is invariant under a Lie group of point transformations, one can find, constructively, special solutions, called *similarity solutions* or *invariant solutions*, that are invariant under a subgroup of the full group admitted by the system. These solutions result from solving a reduced system of differential equations with fewer independent variables. This application of Lie groups was discovered by Lie but first came to prominence in the late 1950s through the work of the Soviet group at Novosibirsk, led by Ovsiannikov (1962, 1982). Invariant solutions can also be constructed for specific boundary value problems. Here one seeks a subgroup of the full group of a given PDE that leaves invariant the boundary curves and the conditions imposed on them [Bluman and Cole (1974)]. Such solutions include *self-similar* (*automodel*) *solutions* that can be obtained through *dimensional analysis* or, more generally, from invariance under groups of scalings. Connections between invariant solutions and separation of variables have been studied extensively by Miller (1977) and coworkers. For ODEs, invariant solutions have particularly nice geometrical properties and include separatrices and envelope solutions [Bluman (1990c); Dresner (1999)].