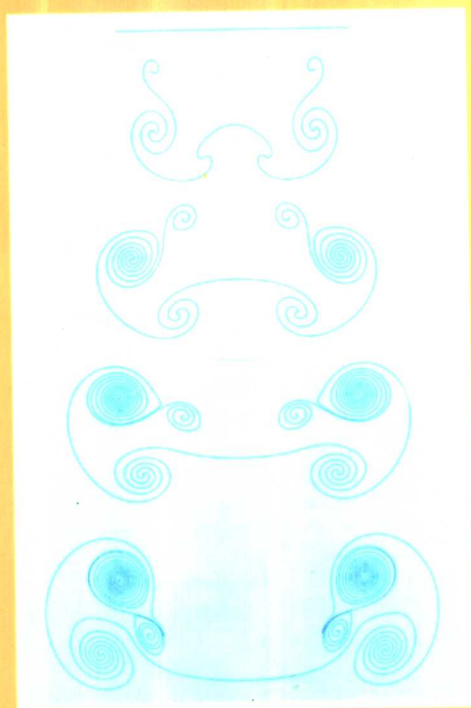


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Vorticity and Incompressible Flow

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ANDREW J. MAJDA
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Vorticity and Incompressible Flow

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CAMBRIDGE
UNIVERSITY PRESS

世界图书出版公司

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
10 Stamford Road, Oakleigh, VIC 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

<http://www.cambridge.org>

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First published 2002

Typeface Times Roman 10/13 pt. *System* L^AT_EX 2_ε [TB]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data

Majda, Andrew, 1949–

Vorticity and incompressible flow / Andrew J. Majda, Andrea L. Bertozzi.

p. cm. – (Cambridge texts in applied mathematics ; 27)

Includes bibliographical references and index.

ISBN 0-521-63057-6 (hb) – ISBN 0-521-63948-4 (pb)

1. Vortex-motion. 2. Non-Newtonian fluids. I. Bertozzi, Andrea L. II. Title. III. Series.

QA925 .M35 2001

532'.059 – dc21

00-046776

ISBN 0 521 63057 6 hardback

ISBN 0 521 63948 4 paperback

This edition of Vorticity and Incompressible Flow by A. J. Majda and A. L. Bertozzi is published by arrangement with the Syndicate of the Press of University of Cambridge, Cambridge, England.

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Not for export elsewhere.

书 名: Vorticity and Incompressible Flow
作 者: A. J. Majda, A. L. Bertozzi
中译名: 涡和不可压缩流
出 版 者: 世界图书出版公司北京公司
印 刷 者: 北京世图印刷厂
发 行: 世界图书出版公司北京公司 (北京朝内大街 137 号 100010)
联系电话: 010-64015659, 64038347
电子信箱: kjsk@vip.sina.com
开 本: 24 印 张: 24
出版年代: 2003 年 9 月
书 号: 7-5062-6553-2 / O · 415
版权登记: 图字: 01-2003-6242
定 价: 98.00 元

世界图书出版公司北京公司已获得 Cambridge University Press 授权在中国大陆
独家重印发行。

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Preface

Vorticity is perhaps the most important facet of turbulent fluid flows. This book is intended to be a comprehensive introduction to the mathematical theory of vorticity and incompressible flow ranging from elementary introductory material to current research topics. Although the contents center on mathematical theory, many parts of the book showcase a modern applied mathematics interaction among rigorous mathematical theory, numerical, asymptotic, and qualitative simplified modeling, and physical phenomena. The interested reader can see many examples of this symbiotic interaction throughout the book, especially in Chaps. 4–9 and 13. The authors hope that this point of view will be interesting to mathematicians as well as other scientists and engineers with interest in the mathematical theory of incompressible flows.

The first seven chapters comprise material for an introductory graduate course on vorticity and incompressible flow. Chapters 1 and 2 contain elementary material on incompressible flow, emphasizing the role of vorticity and vortex dynamics together with a review of concepts from partial differential equations that are useful elsewhere in the book. These formulations of the equations of motion for incompressible flow are utilized in Chaps. 3 and 4 to study the existence of solutions, accumulation of vorticity, and convergence of numerical approximations through a variety of flexible mathematical techniques. Chapter 5 involves the interplay between mathematical theory and numerical or quantitative modeling in the search for singular solutions to the Euler equations. In Chap. 6, the authors discuss vortex methods as numerical procedures for incompressible flows; here some of the exact solutions from Chaps. 1 and 2 are utilized as simplified models to study numerical methods and their performance on unambiguous test problems. Chapter 7 is an introduction to the novel equations for interacting vortex filaments that emerge from careful asymptotic analysis.

The material in the second part of the book can be used for a graduate course on the theory for weak solutions for incompressible flow with an emphasis on modern applied mathematics. Chapter 8 is an introduction to the mildest weak solutions such as patches of vorticity in which there is a complete and elegant mathematical theory. In contrast, Chap. 9 involves a discussion of subtle theoretical and computational issues involved with vortex sheets as the most singular weak solutions in two-space dimensions with practical significance. This chapter also provides a pedagogical introduction to the mathematical material on weak solutions presented in Chaps. 10–12.

Chapter 13 involves a theoretical and computational study of the one-dimensional Vlasov–Poisson equations, which serves as a simplified model in which many of the unresolved issues for weak solutions of the Euler equations can be answered in an explicit and unambiguous fashion.

This book is a direct outgrowth of several extensive lecture courses by Majda on these topics at Princeton University during 1985, 1988, 1990, and 1993, and at the Courant Institute in 1995. This material has been supplemented by research expository contributions based on both the authors' work and on other current research.

Andrew Majda would like to thank many former students in these courses who contributed to the write-up of earlier versions of the notes, especially Dongho Chae, Richard Dziurzynski, Richard McLaughlin, David Stuart, and Enrique Thomann. In addition, many friends and scientific collaborators have made explicit or implicit contributions to the material in this book. They include Tom Beale, Alexandre Chorin, Peter Constantin, Rupert Klein, and George Majda. Ron DiPerna was a truly brilliant mathematician and wonderful collaborator who passed away far too early in his life; it is a privilege to give an exposition of aspects of our joint work in the later chapters of this book.

We would also like to thank the following people for their contributions to the development of the manuscript through proofreading and help with the figures and typesetting: Michael Brenner, Richard Clelland, Diego Cordoba, Weinan E, Pedro Embid, Andrew Ferrari, Judy Horowitz, Benjamin Jones, Phyllis Kronhaus, Monika Nitsche, Mary Pugh, Philip Riley, Thomas Witelski, and Yuxi Zheng. We thank Robert Krasny for providing us with Figures 9.4 and 9.5 in Chap. 9.

Vorticity and Incompressible Flow

This book is a comprehensive introduction to the mathematical theory of vorticity and incompressible flow ranging from elementary introductory material to current research topics. Although the contents center on mathematical theory, many parts of the book showcase the interactions among rigorous mathematical theory, numerical, asymptotic, and qualitative simplified modeling, and physical phenomena. The first half forms an introductory graduate course on vorticity and incompressible flow. The second half comprises a modern applied mathematics graduate course on the weak solution theory for incompressible flow.

Andrew J. Majda is the Samuel Morse Professor of Arts and Sciences at the Courant Institute of Mathematical Sciences of New York University. He is a member of the National Academy of Sciences and has received numerous honors and awards including the National Academy of Science Prize in Applied Mathematics, the John von Neumann Prize of the American Mathematical Society and an honorary Ph.D. degree from Purdue University. Majda is well known for both his theoretical contributions to partial differential equations and his applied contributions to diverse areas besides incompressible flow such as scattering theory, shock waves, combustion, vortex motion and turbulent diffusion. His current applied research interests are centered around Atmosphere/Ocean science.

Andrea L. Bertozzi is Professor of Mathematics and Physics at Duke University. She has received several honors including a Sloan Research Fellowship (1995) and the Presidential Early Career Award for Scientists and Engineers (PECASE). Her research accomplishments in addition to incompressible flow include both theoretical and applied contributions to the understanding of thin liquid films and moving contact lines.

Contents

Preface		<i>page</i> xi
1	An Introduction to Vortex Dynamics for Incompressible Fluid Flows	1
1.1	The Euler and the Navier–Stokes Equations	2
1.2	Symmetry Groups for the Euler and the Navier–Stokes Equations	3
1.3	Particle Trajectories	4
1.4	The Vorticity, a Deformation Matrix, and Some Elementary Exact Solutions	6
1.5	Simple Exact Solutions with Convection, Vortex Stretching, and Diffusion	13
1.6	Some Remarkable Properties of the Vorticity in Ideal Fluid Flows	20
1.7	Conserved Quantities in Ideal and Viscous Fluid Flows	24
1.8	Leray’s Formulation of Incompressible Flows and Hodge’s Decomposition of Vector Fields	30
1.9	Appendix	35
	Notes for Chapter 1	41
	References for Chapter 1	42
2	The Vorticity-Stream Formulation of the Euler and the Navier-Stokes Equations	43
2.1	The Vorticity-Stream Formulation for 2D Flows	44
2.2	A General Method for Constructing Exact Steady Solutions to the 2D Euler Equations	46
2.3	Some Special 3D Flows with Nontrivial Vortex Dynamics	54
2.4	The Vorticity-Stream Formulation for 3D Flows	70
2.5	Formulation of the Euler Equation as an Integrodifferential Equation for the Particle Trajectories	81
	Notes for Chapter 2	84
	References for Chapter 2	84
3	Energy Methods for the Euler and the Navier–Stokes Equations	86
3.1	Energy Methods: Elementary Concepts	87

3.2	Local-in-Time Existence of Solutions by Means of Energy Methods	96
3.3	Accumulation of Vorticity and the Existence of Smooth Solutions Globally in Time	114
3.4	Viscous-Splitting Algorithms for the Navier–Stokes Equation	119
3.5	Appendix for Chapter 3	129
	Notes for Chapter 3	133
	References for Chapter 3	134
4	The Particle-Trajectory Method for Existence and Uniqueness of Solutions to the Euler Equation	136
4.1	The Local-in-Time Existence of Inviscid Solutions	138
4.2	Link between Global-in-Time Existence of Smooth Solutions and the Accumulation of Vorticity through Stretching	146
4.3	Global Existence of 3D Axisymmetric Flows without Swirl	152
4.4	Higher Regularity	155
4.5	Appendixes for Chapter 4	158
	Notes for Chapter 4	166
	References for Chapter 4	167
5	The Search for Singular Solutions to the 3D Euler Equations	168
5.1	The Interplay between Mathematical Theory and Numerical Computations in the Search for Singular Solutions	170
5.2	A Simple 1D Model for the 3D Vorticity Equation	173
5.3	A 2D Model for Potential Singularity Formation in 3D Euler Equations	177
5.4	Potential Singularities in 3D Axisymmetric Flows with Swirl	185
5.5	Do the 3D Euler Solutions Become Singular in Finite Times?	187
	Notes for Chapter 5	188
	References for Chapter 5	188
6	Computational Vortex Methods	190
6.1	The Random-Vortex Method for Viscous Strained Shear Layers	192
6.2	2D Inviscid Vortex Methods	208
6.3	3D Inviscid-Vortex Methods	211
6.4	Convergence of Inviscid-Vortex Methods	216
6.5	Computational Performance of the 2D Inviscid-Vortex Method on a Simple Model Problem	227
6.6	The Random-Vortex Method in Two Dimensions	232
6.7	Appendix for Chapter 6	247
	Notes for Chapter 6	253
	References for Chapter 6	254
7	Simplified Asymptotic Equations for Slender Vortex Filaments	256
7.1	The Self-Induction Approximation, Hasimoto's Transform, and the Nonlinear Schrödinger Equation	257

7.2	Simplified Asymptotic Equations with Self-Stretch for a Single Vortex Filament	262
7.3	Interacting Parallel Vortex Filaments – Point Vortices in the Plane	278
7.4	Asymptotic Equations for the Interaction of Nearly Parallel Vortex Filaments	281
7.5	Mathematical and Applied Mathematical Problems Regarding Asymptotic Vortex Filaments	300
	Notes for Chapter 7	301
	References for Chapter 7	301
8	Weak Solutions to the 2D Euler Equations with Initial Vorticity in L^∞	303
8.1	Elliptical Vortices	304
8.2	Weak L^∞ Solutions to the Vorticity Equation	309
8.3	Vortex Patches	329
8.4	Appendix for Chapter 8	354
	Notes for Chapter 8	356
	References for Chapter 8	356
9	Introduction to Vortex Sheets, Weak Solutions, and Approximate-Solution Sequences for the Euler Equation	359
9.1	Weak Formulation of the Euler Equation in Primitive-Variable Form	361
9.2	Classical Vortex Sheets and the Birkhoff–Rott Equation	363
9.3	The Kelvin–Helmholtz Instability	367
9.4	Computing Vortex Sheets	370
9.5	The Development of Oscillations and Concentrations	375
	Notes for Chapter 9	380
	References for Chapter 9	380
10	Weak Solutions and Solution Sequences in Two Dimensions	383
10.1	Approximate-Solution Sequences for the Euler and the Navier–Stokes Equations	385
10.2	Convergence Results for 2D Sequences with L^1 and L^p Vorticity Control	396
	Notes for Chapter 10	403
	References for Chapter 10	403
11	The 2D Euler Equation: Concentrations and Weak Solutions with Vortex-Sheet Initial Data	405
11.1	Weak-* and Reduced Defect Measures	409
11.2	Examples with Concentration	411
11.3	The Vorticity Maximal Function: Decay Rates and Strong Convergence	421
11.4	Existence of Weak Solutions with Vortex-Sheet Initial Data of Distinguished Sign	432
	Notes for Chapter 11	448
	References for Chapter 11	448

12	Reduced Hausdorff Dimension, Oscillations, and Measure-Valued Solutions of the Euler Equations in Two and Three Dimensions	450
12.1	The Reduced Hausdorff Dimension	452
12.2	Oscillations for Approximate-Solution Sequences without L^1 Vorticity Control	472
12.3	Young Measures and Measure-Valued Solutions of the Euler Equations	479
12.4	Measure-Valued Solutions with Oscillations and Concentrations	492
	Notes for Chapter 12	496
	References for Chapter 12	496
13	The Vlasov–Poisson Equations as an Analogy to the Euler Equations for the Study of Weak Solutions	498
13.1	The Analogy between the 2D Euler Equations and the 1D Vlasov–Poisson Equations	502
13.2	The Single-Component 1D Vlasov–Poisson Equation	511
13.3	The Two-Component Vlasov–Poisson System	524
	Note for Chapter 13	541
	References for Chapter 13	541
	Index	543

An Introduction to Vortex Dynamics for Incompressible Fluid Flows

In this book we study incompressible high Reynolds numbers and incompressible inviscid flows. An important aspect of such fluids is that of *vortex dynamics*, which in lay terms refers to the interaction of local swirls or eddies in the fluid. Mathematically we analyze this behavior by studying the rotation or *curl* of the velocity field, called the *vorticity*. In this chapter we introduce the Euler and the Navier–Stokes equations for incompressible fluids and present elementary properties of the equations. We also introduce some elementary examples that both illustrate the kind of phenomena observed in hydrodynamics and function as building blocks for more complicated solutions studied in later chapters of this book.

This chapter is organized as follows. In Section 1.1 we introduce the equations, relevant physical quantities, and notation. Section 1.2 presents basic symmetry groups of the Euler and the Navier–Stokes equations. In Section 1.3 we discuss the motion of a particle that is carried with the fluid. We show that the particle-trajectory map leads to a natural formulation of how quantities evolve with the fluid. Section 1.4 shows how locally an incompressible field can be approximately decomposed into translation, rotation, and deformation components. By means of exact solutions, we show how these simple motions interact in solutions to the Euler or the Navier–Stokes equations. Continuing in this fashion, Section 1.5 examines exact solutions with shear, vorticity, convection, and diffusion. We show that although deformation can increase vorticity, diffusion can balance this effect. Inviscid fluids have the remarkable property that vorticity is transported (and sometimes stretched) along streamlines. We discuss this in detail in Section 1.6, including the fact that vortex lines move with the fluid and circulation over a closed curve is conserved. This is an example of a quantity that is locally conserved. In Section 1.7 we present a number of global quantities, involving spatial integrals of functions of the solution, such as the kinetic energy, velocity, and vorticity flux, that are conserved for the Euler equation. In the case of Navier–Stokes equations, diffusion causes some of these quantities to dissipate. Finally, in Section 1.8, we show that the incompressibility condition leads to a natural reformulation of the equations (which are due to Leray) in which the pressure term can be replaced with a nonlocal bilinear function of the velocity field. This is the sense in which the pressure plays the role of a Lagrange multiplier in the evolution equation. The appendix of this chapter reviews the Fourier series and the Fourier transform

(Subsection 1.9.1), elementary properties of the Poisson equation (Subsection 1.9.2), and elementary properties of the heat equation (Subsection 1.9.3).

1.1. The Euler and the Navier–Stokes Equations

Incompressible flows of homogeneous fluids in all of space $\mathbb{R}^N$, $N = 2, 3$, are solutions of the system of equations

$$\frac{Dv}{Dt} = -\nabla p + \nu \Delta v, \quad (1.1)$$

$$\operatorname{div} v = 0, \quad (x, t) \in \mathbb{R}^N \times [0, \infty), \quad (1.2)$$

$$v|_{t=0} = v_0, \quad x \in \mathbb{R}^N, \quad (1.3)$$

where $v(x, t) \equiv (v^1, v^2, \dots, v^N)^t$ is the fluid velocity, $p(x, t)$ is the scalar pressure, D/Dt is the convective derivative (i.e., the derivative along particle trajectories),

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_{j=1}^N v^j \frac{\partial}{\partial x_j}, \quad (1.4)$$

and div is the divergence of a vector field,

$$\operatorname{div} v = \sum_{j=1}^N \frac{\partial v^j}{\partial x_j}. \quad (1.5)$$

The gradient operator ∇ is

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_N} \right)^t, \quad (1.6)$$

and the Laplace operator Δ is

$$\Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2}. \quad (1.7)$$

A given kinematic constant viscosity $\nu \geq 0$ can be viewed as the reciprocal of the Reynolds number R_ν . For $\nu > 0$, Eq. (1.1) is called the *Navier–Stokes equation*; for $\nu = 0$ it reduces to the *Euler equation*. These equations follow from the conservation of momentum for a continuum (see, e.g., Chorin and Marsden, 1993). Equation (1.2) expresses the incompressibility of the fluid (see Proposition 1.4). The initial value problem [Eqs. (1.1)–(1.3)] is unusual because it contains the time derivatives of only three out of the four unknown functions. In Section 1.8 we show that the pressure $p(x, t)$ plays the role of a Lagrange multiplier and that a nonlocal operator in $\mathbb{R}^N$ determines the pressure from the velocity $v(x, t)$.

This book often considers examples of incompressible fluid flows in the *periodic* case, i.e.,

$$v(x + e_i, t) = v(x, t), \quad i = 1, 2, \dots, N, \quad (1.8)$$

for all x and $t \geq 0$, where e_i are the standard basis vectors in $\mathbb{R}^N$, $e_1 = (1, 0, \dots)'$, etc. Periodic flows provide prototypical examples for fluid flows in bounded domains $\Omega \subset \mathbb{R}^N$. In this case the bounded domain Ω is the N -dimensional torus T^N . Flows on the torus serve as especially good elementary examples because we have Fourier series techniques (see Subsection 1.9.1) for computing explicit solutions. We make use of these methods, e.g., in Proposition 1.18 (the Hodge decomposition of T^N) in this chapter and repeatedly throughout this book.

In many applications, e.g., predicting hurricane paths or controlling large vortices shed by jumbo jets, the viscosity ν is very small: $\nu \sim 10^{-6} - 10^{-3}$. Thus we might anticipate that the behavior of inviscid solutions (with $\nu = 0$) would give a lot of insight into the behavior of viscous solutions for a small viscosity $\nu \ll 1$. In this chapter and Chap. 2 we show this to be true for explicit examples. In Chap. 3 we prove this result for general solutions to the Navier–Stokes equation in $\mathbb{R}^N$ (see Proposition (3.2)).

1.2. Symmetry Groups for the Euler and the Navier–Stokes Equations

Here we list some elementary symmetry groups for solutions to the Euler and the Navier–Stokes equations. By straightforward inspection we get the following proposition.

Proposition 1.1. *Symmetry Groups of the Euler and the Navier–Stokes Equations. Let v, p be a solution to the Euler or the Navier–Stokes equations. Then the following transformations also yield solutions:*

(i) *Galilean invariance: For any constant-velocity vector $c \in \mathbb{R}^N$,*

$$\begin{aligned} v_c(x, t) &= v(x - ct, t) + c, \\ p_c(x, t) &= p(x - ct, t) \end{aligned} \quad (1.9)$$

is also a solution pair.

(ii) *Rotation symmetry: for any rotation matrix Q ($Q^t = Q^{-1}$),*

$$\begin{aligned} v_Q(x, t) &= Q^t v(Qx, t), \\ p_Q(x, t) &= p(Qx, t) \end{aligned} \quad (1.10)$$

is also a solution pair.

(iii) *Scale invariance: for any $\lambda, \tau \in \mathbb{R}$,*

$$v_{\lambda, \tau}(x, t) = \frac{\lambda}{\tau} v\left(\frac{x}{\lambda}, \frac{t}{\tau}\right), \quad p_{\lambda, \tau}(x, t) = \frac{\lambda^2}{\tau^2} p\left(\frac{x}{\lambda}, \frac{t}{\tau}\right), \quad (1.11)$$

is a solution pair to the Euler equation, and for any $\tau \in \mathbb{R}^+$,

$$v_\tau(x, t) = \tau^{-1/2} v\left(\frac{x}{\tau^{1/2}}, \frac{t}{\tau}\right), \quad p_\tau(x, t) = \tau^{-1} p\left(\frac{x}{\tau^{1/2}}, \frac{t}{\tau}\right), \quad (1.12)$$

is a solution pair to the Navier–Stokes equation.

We note that scaling transformations determine the two-parameter symmetry group given in Eqs. (1.11) for the Euler equation. The introduction of viscosity $\nu > 0$, however, restricts this symmetry group to the one-parameter group given in Eqs. (1.12) for the Navier–Stokes equation.

1.3. Particle Trajectories

An important construction used throughout this book is the *particle-trajectory mapping* $X(\cdot, t): \alpha \in \mathbb{R}^N \rightarrow X(\alpha, t) \in \mathbb{R}^N$. Given a fluid velocity $v(x, t)$, $X(\alpha, t) = (X_1, X_2, \dots, X_N)^t$ is the location at time t of a fluid particle initially placed at the point $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)^t$ at time $t = 0$. The following nonlinear ordinary differential equation (ODE) defines particle-trajectory mapping:

$$\frac{dX}{dt}(\alpha, t) = v(X(\alpha, t), t), \quad X(\alpha, 0) = \alpha. \quad (1.13)$$

The parameter α is called the Lagrangian particle marker. The particle-trajectory mapping X has a useful interpretation: An initial domain $\Omega \subset \mathbb{R}^N$ in a fluid evolves in time to $X(\Omega, t) = \{X(\alpha, t): \alpha \in \Omega\}$, with the vector v tangent to the particle trajectory (see Fig. 1.1).

Next we review some elementary properties of $X(\cdot, t)$. We define the Jacobian of this transformation by

$$J(\alpha, t) = \det(\nabla_\alpha X(\alpha, t)). \quad (1.14)$$

We use subscripts to denote partial derivatives and variables of differential operators, e.g., $f_i = \partial/\partial t f$, $\nabla_\alpha = [(\partial/\partial \alpha_1), \dots, (\partial/\partial \alpha_N)]$. The time evolution of the Jacobian J satisfies the following proposition.

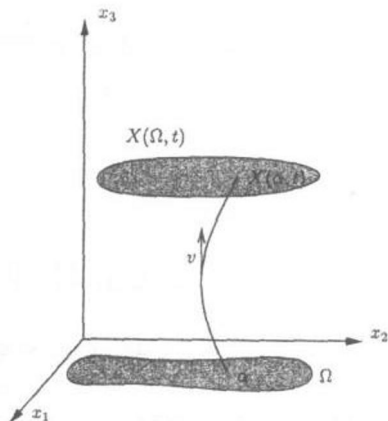


Figure 1.1. The particle-trajectory map.

Proposition 1.2. Let $X(\cdot, t)$ be a particle-trajectory mapping of a smooth velocity field $v \in \mathbb{R}^N$. Then

$$\frac{\partial J}{\partial t} = (\operatorname{div}_x v)|_{(X(\alpha, t), t)} J(\alpha, t). \quad (1.15)$$

We also frequently need a formula to determine the rate of change of a given function $f(x, t)$ in a domain $X(\Omega, t)$ moving with the fluid. This calculus formula, called the transport formula, is the following proposition.

Proposition 1.3. (The Transport Formula). Let $\Omega \subset \mathbb{R}^N$ be an open, bounded domain with a smooth boundary, and let X be a given particle-trajectory mapping of a smooth velocity field v . Then for any smooth function $f(x, t)$,

$$\frac{d}{dt} \int_{X(\Omega, t)} f \, dx = \int_{X(\Omega, t)} [f_t + \operatorname{div}_x(fv)] \, dx. \quad (1.16)$$

We give the proofs of Propositions 1.2 and 1.3 below. As an immediate application of these results, we note that either $J(\alpha, t) = 1$ or $\operatorname{div} v = 0$ implies incompressibility.

Definition 1.1. A flow $X(\cdot, t)$ is incompressible if for all subdomains Ω with smooth boundaries and any $t > 0$ the flow is volume preserving:

$$\operatorname{vol} X(\Omega, t) = \operatorname{vol} \Omega.$$

Applying the transport formula in Eq. (1.16) for $f \equiv 1$, we get $\operatorname{div} v = 0$. Moreover, then Eq. (1.15) yields $J(\alpha, t) = J(\alpha, 0) = 1$. We state this as a proposition below.

Proposition 1.4. For smooth flows the following three conditions are equivalent:

- (i) a flow is incompressible, i.e., $\forall \Omega \subset \mathbb{R}^N, t \geq 0 \operatorname{vol} X(\Omega, t) = \operatorname{vol} \Omega$,
- (ii) $\operatorname{div} v = 0$,
- (iii) $J(\alpha, t) = 1$.

Now we give the proof of Proposition 1.2.

Proof of Proposition 1.2. Because the determinant is multilinear in columns (rows), we compute the time derivative

$$\frac{\partial J}{\partial t} = \frac{\partial}{\partial t} \det \left[\frac{\partial X^i}{\partial \alpha_j}(\alpha, t) \right] = \sum_{i,j} A_i^j \frac{\partial}{\partial t} \frac{\partial X^i}{\partial \alpha_j}(\alpha, t),$$

where A_i^j is the minor of the element $\partial X^i / \partial \alpha_j$ of the matrix $\nabla_\alpha X$. The minors satisfy the well-known identity

$$\sum_j \frac{\partial X^k}{\partial \alpha_j} A_i^j = \delta_i^k J, \quad \text{where } \delta_i^k = \begin{cases} 1, & k = i \\ 0, & k \neq i \end{cases}.$$