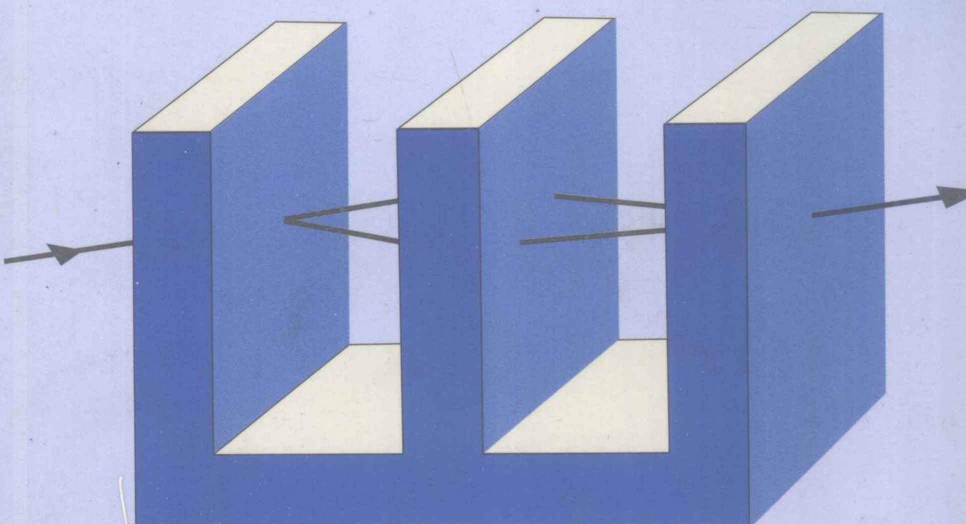
G. L. Squires

Problems in quantum mechanics

with solutions

量子力学题解



CAMBRIDGE
世界图书出版公司

Problems in quantum mechanics

with solutions

G. L. SQUIRES

Emeritus Lecturer in Physics at the University of Cambridge and Fellow of Trinity College, Cambridge

CAMBRIDGE
UNIVERSITY PRESS

世界图书出版公司

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge CB2 1RP, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, United Kingdom
40 West 20th Street, New York, NY 10011-4211, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1995
Reprinted 1995, 1996

Printed in the United Kingdom at the University Press, Cambridge

Typeset in Linotron Times 10/12.5 pt

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing in Publication Data
Squires, G. L. (Gordon Leslie)
Problems in quantum mechanics with solutions / G. L. Squires.
p. cm.
Includes bibliographical references and index.
ISBN 0-521-37245-3. – ISBN 0-521-37850-8 (pbk.)
1. Quantum theory – Problems, exercises, etc. I. Title.
QC174. 15. S66 1995
530. 1'2 – dc20 93-43931 CIP

ISBN 0 521 37245 3 hardback
ISBN 0 521 37850 8 paperback

This edition of Problems in Quantum Mechanics by Gordon Squires is published by arrangement with the Syndicate of the Press of University of Cambridge, Cambridge, England.

Problem solving in physics is not simply a test of understanding the subject, but is an integral part of learning it. In this book, the basic ideas and methods of quantum mechanics are illustrated by means of a carefully chosen set of problems, complete with detailed, step-by-step solutions.

After a preliminary chapter on orders of magnitude, a variety of topics are covered, including the postulates of quantum mechanics, Schrödinger's equation, angular momentum, the hydrogen atom, the harmonic oscillator, spin, time-independent and time-dependent perturbation theory, the variational method, identical particles, multielectron atoms, transitions and scattering. Most of the chapters start with a summary of the relevant theory, outlining the required background for a given group of problems. Considerable emphasis is placed on examples from atomic, solid-state and nuclear physics, particularly in the latter part of the book as the student's familiarity with the concepts and techniques increases.

Throughout, the physical interpretation or application of the results is highlighted, thereby providing useful insights into a wide range of systems and phenomena. This approach will make the book invaluable to anyone taking an undergraduate course in quantum mechanics.

Acknowledgements

I started this book in 1969. I took sabbatical leave in 1970, at the Hebrew University in Jerusalem, intending to finish it there. However I was side-tracked by meeting the woman whom I subsequently married. So, like P. G. Wodehouse, I dedicate this book to my wife, but for whom it would have been written twenty three years ago.

I wish to thank Dr K. F. Riley, and Messrs S. R. Johnson, S. Patel, and B. E. Rafferty who read parts of the manuscript and made several useful comments on it. I am particularly grateful to Dr M. E. Cates and Messrs D. M. Freye, F. M. Grosche, R. K. W. Haselwimmer, R. J. F. Hughes, and C. S. Reynolds, who between them worked through the problems and made valuable comments and suggestions. Finally, I wish to acknowledge my indebtedness to all the undergraduates, mainly from Trinity College, whom I have supervised in Quantum Mechanics in the last thirty years. In the words of the proverb 'I have learnt much from my teachers, but more from my pupils.'

G. L. Squires

Preface for the reader

The problems in this book are intended to cover the topics in an average second- and third-year undergraduate course in Quantum Mechanics. After a preliminary chapter on orders of magnitude, there are eight chapters on topics arranged in a fairly conventional order. The tenth and final chapter contains a selection of miscellaneous problems on the topics of the previous chapters. I have separated them from the others on the grounds of their being somewhat longer and perhaps more difficult. But you should not be deterred from trying them on that account.

The important thing for all the problems is that you *do attempt them*. If you attempt a problem, and think about it, but cannot solve it, and *then* look up the solution, you will get much more benefit than if you jump to the solution as soon as you have read the problem. If you can solve a problem, you are still advised to look at the solution, which might contain a quicker or neater method than the one you have used. (If yours is quicker or neater I shall be pleased to hear from you.) I have also included some comments at the ends of some of the solutions, which you may find useful. They relate, either to the algebraic technique, or, more commonly, to a physical interpretation or application of the result.

At the beginnings of Chapters 2 to 9, I have included sections entitled Summary of theory, and you should read the summary before trying the problems in the chapter. The summary has a two-fold object. One is to introduce the notation, and the other is to inform you what you need to know before you attempt the problems. The results are quoted without proofs, which it is assumed you will obtain in your lecture course.

The equations are numbered independently in each solution and summary. A single equation number refers to the equation within the current solution or summary. An equation in another solution is referred to by a triple number, e.g. (5.7.3) is equation 3 in the solution to Problem 5.7. Reference from outside to an equation in a summary is made by a double number, so (4.8) is equation 8 in the summary for Chapter 4.

Contents

1
Numerical values

Values of physical constants

speed of light	c		$= 2.998 \times 10^{8}\ \mathrm{m\,s^{-1}}$
permittivity of vacuum	ε_0	$= 1/\mu_0 c^2$	$= 8.854 \times 10^{-12}\ \mathrm{F\,m^{-1}}$
Planck constant	h		$= 6.626 \times 10^{-34}\ \mathrm{J\,s}$
	$\hbar$	$= h/2\pi$	$= 1.055 \times 10^{-34}\ \mathrm{J\,s}$
elementary charge	e		$= 1.602 \times 10^{-19}\ \mathrm{C}$
Boltzmann constant	k_B		$= 1.381 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Avogadro constant	N_A		$= 6.022 \times 10^{23}\ \mathrm{mol^{-1}}$
mass of electron	m_e		$= 9.109 \times 10^{-31}\ \mathrm{kg}$
mass of proton	m_p		$= 1.673 \times 10^{-27}\ \mathrm{kg}$
mass of neutron	m_n		$= 1.675 \times 10^{-27}\ \mathrm{kg}$
atomic mass unit	m_u	$= 10^{-3}/N_A$	$= 1.661 \times 10^{-27}\ \mathrm{kg}$
Bohr radius	a_0	$= 4\pi\varepsilon_0\hbar^2/e^2 m_e$	$= 5.292 \times 10^{-11}\ \mathrm{m}$
Rydberg constant	R_∞	$= \hbar/4\pi c m_e a_0^2$	$= 1.097 \times 10^{7}\ \mathrm{m^{-1}}$
fine structure constant	α	$= e^2/4\pi\varepsilon_0 c\hbar$	$= 7.297 \times 10^{-3}$
Bohr magneton	μ_B	$= e\hbar/2m_e$	$= 9.274 \times 10^{-24}\ \mathrm{J\,T^{-1}}$
nuclear magneton	μ_N	$= e\hbar/2m_p$	$= 5.051 \times 10^{-27}\ \mathrm{J\,T^{-1}}$

The values of the physical constants given above are sufficiently precise for the calculations in the present book. In fact, these constants are known with a fractional error of 10^{-6} or less, apart from the Boltzmann constant where the fractional error is about 10^{-5}. A list of the values of the fundamental physical constants which are the best fit to the results of a variety of precision measurements has been prepared by Cohen and Taylor (1986).

Problems

The values of physical constants are given on p. 1. The answers to Problems 1.1 to 1.7 should be given to 3 significant digits.

1.1 The ionisation energy of the hydrogen atom in its ground state is $E_{\text{ion}} = 13.60\ \text{eV}$. Calculate the frequency, wavelength, and wave number of the electromagnetic radiation that will just ionise the atom.

1.2 Atomic clocks are so stable the second is now *defined* as the duration of 9 192 631 770 periods of oscillation of the radiation corresponding to the transition between two closely spaced energy levels in the caesium-133 atom. Calculate the energy difference between the two levels in eV.

1.3 A He–Ne laser emits radiation with wavelength $\lambda = 633\ \text{nm}$. How many photons are emitted per second by a laser with a power of 1 mW?

1.4 In the presence of a nucleus, the energy of a γ-ray photon can be converted into an electron–positron pair. Calculate the minimum energy of the photon in MeV for this process to occur. What is the frequency corresponding to this energy?
[The mass of the positron is equal to that of the electron.]

1.5 If a dc potential V is applied across two layers of superconducting material separated by a thin insulating barrier, an oscillating current of paired electrons passes between them by a tunnelling process. The frequency ν of the oscillation is given by $h\nu = 2eV$. Calculate the value of ν when a potential of 1 V is applied across the two superconductors.

1.6 (a) The magnetic dipole moment $\boldsymbol{\mu}$ of a current loop is defined by

$$\boldsymbol{\mu} = I\mathbf{A},$$

where I is the current, and $\mathbf{A}$ is the area of the loop, the direction of $\mathbf{A}$ being perpendicular to the plane of the loop. A current loop may be represented by a charge e rotating at constant speed in a small circular orbit. Use classical reasoning to show that the magnetic dipole moment of the loop is related to $\mathbf{L}$, the orbital angular momentum of the particle, by

$$\boldsymbol{\mu} = \frac{e}{2m}\mathbf{L},$$

where m is the mass of the particle.

(b) If the magnitude of **L** is $\hbar$ ($= h/2\pi$), calculate the magnitude of $\boldsymbol{\mu}$ for (i) an electron, and (ii) a proton.

1.7 Calculate the value of the magnetic field required to maintain a stream of protons of energy 1 MeV in a circular orbit of radius 100 mm.

1.8 Neutron diffraction may be used to determine crystal structures.

(a) Estimate a suitable value for the velocity of the neutrons.

(b) Calculate the kinetic energy of the neutron in eV for this velocity.

(c) It is common practice in this type of experiment to select a beam of monoenergetic neutrons from a gas of neutrons at temperature T. Estimate a suitable value for T.

1.9 The most accurate values of the sizes of atomic nuclei come from measurements of electron scattering. Estimate roughly the energies of electrons that provide useful information.

Solutions

1.1 The ionisation energy of hydrogen in the ground state is

$$E_{\text{ion}} = 13.60\,\text{eV} = 2.18 \times 10^{-18}\,\text{J}. \tag{1}$$

The frequency of the radiation that will just ionise the atom is

$$\nu = \frac{E_{\text{ion}}}{h} = 3.29 \times 10^{15}\,\text{Hz}. \tag{2}$$

The wavelength λ and wavenumber $\tilde{\nu}$ of the radiation are

$$\lambda = \frac{c}{\nu} = 9.12 \times 10^{-8}\,\text{m}, \tag{3}$$

$$\tilde{\nu} = \frac{1}{\lambda} = 1.10 \times 10^{7}\,\text{m}^{-1}. \tag{4}$$

1.2 The energy difference between the two levels is

$$\Delta E = \frac{h\nu}{e} = 3.80 \times 10^{-5}\,\text{eV}. \tag{1}$$

1.3 The energy of each photon is

$$E = \frac{hc}{\lambda}, \tag{1}$$

where

$$\lambda = 6.33 \times 10^{-7}\,\text{m}. \tag{2}$$

The power of the laser is

$$P = 1\,\text{mW}. \tag{3}$$

The number of photons emitted per second is

$$n = \frac{P}{E} = \frac{P\lambda}{hc} = 3.19 \times 10^{15}. \tag{4}$$

1.4 (a) The minimum energy E_{min} of the γ-ray photon required for the production of an electron and a positron is equal to the sum of the rest mass energies of the two particles. The mass of the positron is equal to m_e, the mass of the electron. So the required value is

$$E_{\min} = \frac{2m_e c^2}{10^6 e} = 1.02\,\text{MeV}. \tag{1}$$

(b) The frequency ν corresponding to this energy is

$$\nu = \frac{2m_e c^2}{h} = 2.47 \times 10^{20}\,\text{Hz}. \tag{2}$$

1.5 The frequency of oscillation ν of the current is given by

$$h\nu = 2eV. \tag{1}$$

For $V = 1$ volt, the frequency is

$$\nu = 4.84 \times 10^{14}\,\text{Hz}. \tag{2}$$

By measuring the frequency we can deduce the value of the applied voltage from (1). The phenomenon provides a high-precision method of measuring a potential difference – see Solution 8.10, Comment (2) on p. 178.

1.6 (a) Denote the radius of the orbit by a, and the speed of the particle by v. Then the period of revolution is $\tau = 2\pi a/v$. The current due to the rotating charge is

$$I = \frac{e}{\tau} = \frac{ev}{2\pi a}. \tag{1}$$

The magnetic dipole moment is

$$\mu = IA = \frac{ev}{2\pi a}\pi a^2 = \tfrac{1}{2}eva. \tag{2}$$

The orbital angular momentum is

$$L = mav. \tag{3}$$

Therefore

$$\boldsymbol{\mu} = \frac{e}{2m}\mathbf{L}. \tag{4}$$

The vector form follows because, for positive e, the quantities $\boldsymbol{\mu}$ and $\mathbf{L}$ are in the same direction.

(b) For $L = \hbar$, the magnetic dipole moment of a circulating electron is

$$\mu_e = \frac{e\hbar}{2m_e} = 9.28 \times 10^{-24}\,\text{J}\,\text{T}^{-1}, \tag{5}$$

while, for a circulating proton, it is

$$\mu_p = \frac{e\hbar}{2m_p} = 5.05 \times 10^{-27}\,\mathrm{J\,T^{-1}}. \tag{6}$$

Comments

Although the result in (4) has been derived by classical reasoning for the special case of a charge moving in a circular orbit, it is valid for orbital motion in general in quantum mechanics. A particle, such as an electron or a proton, in a stationary state does not move in a definite orbit, nor does it have a definite speed, but it does have a definite orbital angular momentum, the component of which in any direction is of the form $n\hbar$, where n is an integer, positive, negative, or zero. Thus $\hbar$ may be regarded as a natural unit of angular momentum. Since magnetic dipole moment and angular momentum are related by (4), the component of magnetic dipole moment of an electron, due to its orbital motion, has the form $n\mu_B$, where

$$\mu_B = \frac{e\hbar}{2m_e}. \tag{7}$$

Thus μ_B, known as the *Bohr magneton*, is the natural unit of magnetic dipole moment for the electron. Similarly the quantity

$$\mu_N = \frac{e\hbar}{2m_p}, \tag{8}$$

known as a *nuclear magneton*, is the natural unit of magnetic dipole moment for the proton.

The simple relation in (4) between magnetic dipole moment and angular momentum does not apply when the effects of the *intrinsic* or *spin* angular momentum of the particle are taken into account. However, it remains true that the magnetic dipole moments of atoms are of the order of Bohr magnetons, while the magnetic dipole moments of the proton, the neutron, and of nuclei in general, are of the order of nuclear magnetons.

1.7 If the velocity of the proton is **v**, the Lorentz force acting on it, due to the magnetic field **B**, is $e[\mathbf{v} \times \mathbf{B}]$. The force is perpendicular to the instantaneous direction of motion and to the direction of **B**. Thus the protons move in a circle, the plane of which is perpendicular to **B**. Equating the force to the mass times the centripetal acceleration for circular motion, we have

$$Bev = \frac{m_p v^2}{a}, \tag{1}$$

where a is the radius of the circle. Whence

$$B = \frac{m_p v}{ea} = \frac{(2m_p E)^{1/2}}{ea}. \tag{2}$$

Inserting the values of the constants, together with $E = 10^6\,\mathrm{eV} = (10^6 e)\,\mathrm{J}$, and $a = 0.1\,\mathrm{m}$, gives

$$B = 1.45\,\mathrm{T}. \tag{3}$$

1.8 (a) To obtain information on the crystal structure, neutrons are diffracted by the crystal in accordance with Bragg's law

$$n\lambda = 2d \sin\theta. \tag{1}$$

(This is the same law that governs the diffraction of X-rays.) In this equation, λ is the wavelength of the neutrons, d is the distance between the planes of diffracting atoms, θ is the glancing angle between the direction of the incident neutrons and the planes of atoms, and n is an integer (usually small). The equation cannot be satisfied unless $\lambda < 2d$. On the other hand, if $\lambda \ll 2d$, θ is inconveniently small. So it is necessary for λ to be of the same order as d, which is of the order of the interatomic spacing in the crystal. Put $\lambda = d = 0.2\,\mathrm{nm}$ (a typical value).

The de Broglie relation between λ and the velocity v of the neutron is

$$\lambda = \frac{h}{m_n v}, \tag{2}$$

where m_n, the mass of the neutron, is $1.675 \times 10^{-27}\,\mathrm{kg}$. Thus

$$v = \frac{h}{m_n \lambda} = 2.0\,\mathrm{km\,s^{-1}}. \tag{3}$$

(b) The kinetic energy of the neutrons is

$$E = \tfrac{1}{2} m_n v^2 = 3.3 \times 10^{-21}\,\mathrm{J} = 20\,\mathrm{meV} \tag{4}$$

for the above velocity.

(c) Put $E = k_B T$. Then the above value of E corresponds to $T = 240\,\mathrm{K}$, which is of the order of room temperature. Such neutrons are readily available in a thermal nuclear reactor; they are termed *thermal neutrons*.

1.9 The electrons scattered by nuclei show diffraction effects characteristic of the radius r of the nucleus, the value of which lies in the range

1–6 fm (1 fm = 10^{-15} m). As in Problem 1.8, measurable effects require that the wavelength λ of the electron should be of the order of r. Thus the momentum p of the electron should satisfy

$$p = \frac{h}{\lambda} \approx \frac{h}{r} = 1.3 \times 10^{-19}\,\mathrm{kg\,m\,s^{-1}}, \tag{1}$$

for $r = 5$ fm. This value is very much larger than

$$m_e c = 2.7 \times 10^{-22}\,\mathrm{kg\,m\,s^{-1}}, \tag{2}$$

where m_e is the rest mass of the electron, which shows that the electrons required for the measurements are highly relativistic.

The energy E of the electrons is related to their momentum p by

$$E^2 = m_e^2 c^4 + p^2 c^2. \tag{3}$$

Since $p \gg m_e c$, we can neglect the first term on the right-hand side of (3). Thus

$$E \approx pc = 4.0 \times 10^{-11}\,\mathrm{J} = 250\,\mathrm{MeV}. \tag{4}$$

The value obtained for E clearly depends on the value taken for r. If E is in MeV, and λ $(= r)$ is in fm, you may verify that, for the highly relativistic case,

$$E\lambda \approx 10^9 \times \frac{ch}{e} = 1240. \tag{5}$$

2
Fundamentals

Summary of theory

1 What you need to know

Definitions and properties

Operator, linear operator, functions of operators, commuting and non-commuting operators, eigenfunction, eigenvalue, degeneracy, normalised function, orthogonal functions, Hermitian operator.

2 Postulates of quantum mechanics

(1) The state of a system with n position variables q_1, q_2, ... q_n is specified by a state (or wave) function $\psi(q_1, q_2, \ldots q_n)$. All possible information about the system can be derived from this state function. In general, n is three times the number of particles in the system. So for a single particle $n = 3$, and q_1, q_2, q_3 may be the Cartesian coordinates x, y, z, or the spherical polar coordinates r, θ, ϕ, or some other set of coordinates.

(2) To every observable there corresponds a Hermitian operator given by the following rules:

(i) The operator corresponding to the Cartesian position coordinate x is $x \times$ – similarly for the coordinates y and z.

(ii) The operator corresponding to p_x, the x component of linear momentum, is $(\hbar/\mathrm{i})\partial/\partial x$ – similarly for the y and z components.

(iii) To obtain the operator corresponding to any other observable, first write down the classical expression for the observable in terms of x, y, z, p_x, p_y, p_z, and then replace each of these quantities by its corresponding operator according to rules (i) and (ii).

(3) The only possible result which can be obtained when a measurement is made of an observable whose operator is A is an eigenvalue of A.

(4) Let α be an observable whose operator A has a set of eigenfunctions ϕ_j with corresponding eigenvalues a_j. If a large number of

measurements of α are made on a system in the state ψ, then the expectation value of α for the state ψ (i.e. the arithmetic mean of the eigenvalues obtained) is given by

$$\langle A \rangle = \int \psi^* A \psi \, d\tau, \tag{1}$$

where $d\tau$ is an element of volume, and the integral is taken over all space.

(5) If the result of a measurement of α is a_r, corresponding to the eigenfunction ϕ_r, then the state function immediately after the measurement is ϕ_r.

This means that in general a measurement changes or disturbs the state of a system. The set of measurements referred to in the 4th postulate are all made on the system in the same state ψ. It is in general necessary to manipulate the system after each measurement to return it to the state ψ before the next measurement is made.

(6) The time variation of the state function of a system is given by

$$\frac{\partial \psi}{\partial t} = \frac{1}{i\hbar} H \psi, \tag{2}$$

where H is the operator formed from the classical Hamiltonian of the system.

Note on Postulate 2 (iii)
If the classical expression for an observable contains a product $\alpha\beta$ whose operators A and B do not commute, then the operator corresponding to $\alpha\beta$ is $\frac{1}{2}(AB + BA)$. Examples of this are rare.

3 Basic deductions from the postulates

(a) Probability of result of measurement

Discrete eigenvalues. Suppose the eigenvalues a_j of A in postulates 4 and 5 are discrete, and that the state function ψ and all the eigenfunctions ϕ_j of A are normalised. To find the probability p_r that the result of a measurement of the observable α is a particular a_r, expand ψ in terms of the ϕ_j, i.e. put

$$\psi = \sum_j c_j \phi_j. \tag{3}$$

Then

$$p_r = |c_r|^2. \tag{4}$$

The coefficient c_r is obtained from