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Contents.

Analytical Theory of Subsonic and Supersonic Flows.

By

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With 24 Figures.

Introduction.

This article deals with the mathematical theory of motion of a compressible fluid. On account of the non-linear character of the differential equations involved, an analytical treatment of the flow under most general circumstances has not yet been possible. Therefore, in order to facilitate treatment we have throughout made various idealizing assumptions. In particular we neglect viscosity, thermal conductivity and any external forces acting on the fluid and, moreover, we restrict ourselves to stationary irrotational flows. Even with these simplifications the deeper mathematical problems regarding the existence and uniqueness theory have been satisfactorily treated only under the additional requirement of plane flow. And even in this special theory we ran into some unsolved problems of the theory of partial differential equations of mixed elliptic-hyperbolic type which are connected with the transition from subsonic to supersonic flow regime. We have not touched upon the theory of shock discontinuities in the flow as this theory is developed systematically in the article by CABANNES.

The structure of the article is as follows: In the first chapter, we give the basic physical and mathematical concepts of the theory and the fundamental equations of motion. Some explicit solutions illustrate the general theory.

The second chapter deals with the theory of linearized flows. This is a theory of approximations applicable to thin or slender bodies in an otherwise uniform flow. In this case, the fundamental non-linear equations can be replaced by linear differential equations with constant coefficients. Since only a small number of explicit solutions for the correct fundamental equations is known, it is important to possess approximation methods which permit qualitative statements on the nature of the flow considered. The theory of linearization is applicable to flows in the plane and in three-dimensional space and is extensively used in airfoil theory and other branches of applied aerodynamics.

The third chapter is devoted to the hodograph method. This more refined method achieves a rigorous linearization of the differential equations of motion of a compressible fluid. It works, however, only in the case of plane flow and is, moreover, an indirect procedure. Indeed, it yields exact solutions of the correct equations, but not necessarily those solutions which correspond to the desired boundary conditions. The adjustment to the correct side conditions implies approximation procedures which are quite involved. However, at present the hodograph method seems the most promising analytical tool in the theory of plane flows.

The fourth chapter deals with the uniqueness and existence theory of plane subsonic flows. We describe the standard analytical methods of this theory and sketch some fundamental proofs in detail sufficient to allow an understanding of the underlying principles and methods. Existence proofs may appear as rather esoteric to the applied aerodynamist, but a good existence proof contains the germs of a constructive procedure, and the role of the Janzen-Rayleigh iteration is considered in this context. Finally, we describe a method of obtaining inequalities and estimates for the velocity and pressure fields of a flow; it stands in analogy to the distortion theorems in conformal mapping which allow similar estimates in the case of an incompressible flow. When exact solutions are not easily available such estimates may be quite valuable.

The fifth chapter covers the theory of two-dimensional transonic flows. The methods and problems in this theory are described and, in particular, the Taylor problem of the existence of continuous transonic flows is discussed. While the existence of such flows for arbitrary boundaries seems dubious, explicit solutions can be given for special problems. We consider the method of idealized fluids which permits the construction of some transonic flow patterns of practical importance, such as flow through a nozzle.

We have tried to make the different chapters to a large extent independent of each other. In view of the extensive literature in the field, we have aimed less at completeness of results and methods than at a clear description of the basic ideas; for more detail we refer to the general references and textbooks listed at the end of the paper as well as to the bibliography, organized by chapters.

I. Physical and mathematical foundations.

1. Basic assumptions and fundamental equations. The motion of a fluid is described mathematically by the vector field of velocity $\boldsymbol{q}$ and the three scalar fields of pressure p, density ϱ and temperature T. These six quantities are considered as functions of the three space variables x, y, z and the time t.

We shall consider in this article only non-viscous fluids and neglect the influence of external forces like gravity. Then the equations of motion are

$$\varrho \frac{D}{Dt} \boldsymbol{q} = - \nabla p. \tag{1.1}$$

The symbol D/Dt is the Eulerian or hydrodynamical derivative: it denotes differentiation of a field quantity relative to an observer moving with the flow. It takes into account, therefore, the spatial as well as the temporal change and is defined by the well-known operator formula:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\boldsymbol{q} \cdot \nabla). \tag{1.2}$$

We can thus give to (1.1) the alternative form

$$\frac{\partial}{\partial t} \boldsymbol{q} + (\boldsymbol{q} \cdot \nabla) \boldsymbol{q} = - \frac{1}{\varrho} \nabla p. \tag{1.1'}$$

The conservation of matter in the flow leads to another fundamental differential relation between the velocity and the density field, namely, the equation of continuity

$$\frac{\partial}{\partial t} \varrho + \nabla \cdot (\varrho \boldsymbol{q}) = 0. \tag{1.3}$$

The physical nature of the fluid considered enters the theory through the equation of state which connects p, ϱ and T. There are various other thermodynamical quantities like the internal energy per unit of mass U or the entropy per unit mass S. Any two of these variables can be used in order to express all the others in terms of them. In theoretical fluid dynamics it is particularly useful to take the entropy S as one basic variable. This is evident from the definition of the entropy change ΔS with a heat increase ΔQ

$$\Delta S = \frac{\Delta Q}{T} \tag{1.4}$$

and from the first fundamental law of thermodynamics

$$\Delta Q = \Delta U + p \Delta \left(\frac{1}{\varrho}\right). \tag{1.5}$$

It is often permissible to neglect heat conduction in the fluid and we shall do so consistently. We can then state that for a fixed unit of mass

$$\Delta Q = 0 \tag{1.6}$$

throughout the entire flow motion. This guarantees by Eq. (1.4) that the specific entropy of the moving matter remains constant:

$$\frac{DS}{Dt} = \frac{\partial}{\partial t} S + (\boldsymbol{q} \cdot \nabla) S = 0, \tag{1.7}$$

while we see from Eq. (1.5) that T will in general vary during the motion.

A flow which has a constant specific entropy for each moving particle is called an isentropic flow. The theory of most continuous flow phenomena can be carried out under the assumption of isentropy: only in the case of discontinuity surfaces and shock phenomena does the change of entropy play an important role and need to be taken into account.

The six Eqs. (1.1′), (1.3), (1.7) and the equation of state

$$p = f(\varrho, S) \tag{1.8}$$

are a system for the six unknown quantities $\boldsymbol{q}$, ϱ, p, S describing the motion of the fluid. They have to be complemented by proper initial and boundary conditions in order to make the problem of integration a well determined one.

In order to bring the basic equations of motion (1.1′) into a simpler form we make use of the identity of vector analysis

$$(\boldsymbol{q} \cdot \nabla)\, \boldsymbol{q} = \tfrac{1}{2} \nabla (q^2) - \boldsymbol{q} \times (\nabla \times \boldsymbol{q}). \tag{1.9}$$

This identity is useful since it introduces the vector

$$\boldsymbol{\zeta} = \nabla \times \boldsymbol{q} \tag{1.10}$$

which represents the vorticity of the flow and is significant for the flow pattern.

The meaning of the term $\frac{1}{\varrho} \nabla p$ is best understood from the thermodynamical relations (1.4) and (1.5). Since S, U, p and ϱ are well determined functions of the physical state variables in the fluid, we obtain after elimination of ΔQ the generally valid identity:

$$\nabla U = T \nabla S - p \nabla \left(\frac{1}{\varrho}\right). \tag{1.11}$$

We introduce now the specific enthalpy [*1*]

$$H = U + \frac{1}{\varrho} p \tag{1.12}$$

and bring Eq. (1.11) into the form

$$\nabla H = T\,\nabla S + \frac{1}{\varrho}\,\nabla p. \tag{1.11'}$$

Combining Eqs. (1.1'), (1.9) and (1.11'), we then obtain finally

$$\frac{\partial}{\partial t}\boldsymbol{q} + \nabla\left(H + \frac{1}{2}q^2\right) - \boldsymbol{q}\times\boldsymbol{\zeta} = T\,\nabla S. \tag{1.12'}$$

This formulation of the equations of motion is due to CROCCO; it is advantageous since it exhibits clearly the various quantities which have an immediate physical significance [2].

As an illustration of the usefulness of CROCCO's formula we shall derive from it the relation between entropy and vorticity in the flow. Let $C(t)$ be a closed curve moving with the flow, and let it be represented in the parametric form $\boldsymbol{r} = \boldsymbol{r}(s,t)$. The quantity

$$Z(t) = \oint_{C(t)} \boldsymbol{q}\,d\boldsymbol{r} = \int \boldsymbol{q}\,\frac{\partial}{\partial s}\boldsymbol{r}(s,t)\,ds \tag{1.13}$$

is called the circulation around the curve $C(t)$. From Eqs. (1.2), (1.9) and (1.12') we compute

$$\frac{D}{Dt}Z(t) = \int_s \frac{D}{Dt}\boldsymbol{q}\cdot\frac{\partial}{\partial s}\boldsymbol{r}(s,t)\,ds + \int_s \boldsymbol{q}\,\frac{\partial}{\partial s}\boldsymbol{q}\,ds = \oint_{C(t)} T\,\nabla S\cdot d\boldsymbol{r}, \tag{1.14}$$

since $\frac{D}{Dt}\left(\frac{\partial}{\partial s}\boldsymbol{r}\right) = \frac{\partial}{\partial s}\boldsymbol{q}$ on the curve $C(t)$ which moves with the flow and since q^2 and H are single-valued functions in space. The change of circulation is thus closely related to the specific entropy along the curve considered. On the other hand, we have by STOKES' theorem

$$Z = \oint_C \boldsymbol{q}\cdot d\boldsymbol{r} = \iint_\Sigma \boldsymbol{\zeta}\cdot\boldsymbol{n}\,d\sigma, \tag{1.15}$$

where Σ is an arbitrary surface spanned through the curve C and $\boldsymbol{n}$ is the local normal on Σ. This identity and Eq. (1.14) show the close relation between circulation, entropy and vorticity.

A flow in which the fluid has the same constant entropy S is called homentropic. Since in this case $\nabla S \equiv 0$, we have in every homentropic flow the Helmholtz-Kelvin circulation theorem

$$\frac{D}{Dt}Z = 0 \tag{1.16}$$

for every closed curve C.

If a flow is made up of particles which come all from a region with constant entropy it must be homentropic by virtue of the isentropy condition (1.7). If all particles pass through a region where, moreover, the vorticity vector vanishes identically we can conclude from Eqs. (1.16) and (1.15) that the flow has everywhere vorticity zero, that is:

$$\nabla\times\boldsymbol{q} = 0. \tag{1.17}$$

Such a flow is called irrotational.

We shall deal from now on with steady (that is, time independent), irrotational and homentropic flows. Our preceding considerations show that such flows will arise under very general assumptions and are of great importance; on the other hand, the assumptions made will lead to a simple mathematical theory which can be handled without too great complications.

Because of the simplifying assumptions, we are now dealing with the five functions $\boldsymbol{q}, p$ and ϱ which depend only on the three space variables x, y, z. Between p and ϱ there exists the (adiabatic) equation of state

$$p = f(\varrho). \tag{1.18}$$

CROCCO's equation (1.12) reduces to

$$H + \tfrac{1}{2} q^2 = \text{const} \tag{1.19}$$

and the continuity equation has now the form

$$\nabla(\varrho \cdot \boldsymbol{q}) = 0. \tag{1.20}$$

In the homentropic case, Eq. (1.11′) simplifies to

$$H = H(\varrho) = \int \frac{dp}{\varrho} \tag{1.11''}$$

and (1.19) becomes the classical Bernoulli equation

$$\frac{1}{2} q^2 + \int \frac{dp}{\varrho} = \text{const}, \tag{1.19'}$$

which establishes the speed-density relation.

From the condition of irrotationality (1.17) follows the existence of a function $\varphi(x, y, z)$ such that

$$\boldsymbol{q} = \nabla \varphi. \tag{1.21}$$

φ is called the velocity potential of the flow. If we insert Eq. (1.21) into the continuity equation (1.20), we obtain the second order partial differential equation for the velocity potential

$$\nabla(\varrho \nabla \varphi) = 0. \tag{1.22}$$

The mathematical theory of a steady, homentropic and irrotational flow reduces, therefore, to the following procedure: (a) From the adiabatic equation of state (1.18) we compute the enthalpy function (1.11″). (b) By means of BERNOULLI's equation, we express the density as a function of the local speed

$$\varrho = P\big((\nabla \varphi)^2\big) = P(q^2). \tag{1.23}$$

(c) We insert Eq. (1.23) into Eq. (1.22) and obtain the partial differential equation

$$\nabla\big(P((\nabla \varphi)^2) \cdot \nabla \varphi\big) = 0 \tag{1.24}$$

which is to be integrated in accordance with specified boundary conditions.

The entire analytic theory of steady homentropic and irrotational flow is the theory of the non-linear partial differential equation (1.24) and its boundary value problems.

Because of the great importance of Eq. (1.24) for the whole theory we write it out in detail:

$$\varrho \nabla^2 \varphi + \frac{\partial \varrho}{\partial q^2} \cdot \nabla\big((\nabla \varphi)^2\big) \cdot \nabla \varphi = 0. \tag{1.25}$$

From Eq. (1.19′) we deduce

$$\frac{\partial \varrho}{\partial q^2} = -\frac{1}{2} \frac{\varrho}{c^2}, \qquad c^2 = \frac{dp}{d\varrho}. \tag{1.26}$$

The quantity $c = \sqrt{p'(\varrho)}$ is called the local speed of sound since it is the speed at which small perturbations travel, as is apparent from the linearized theory discussed in Chap. II. By means of Eq. (1.26), we obtain for Eq. (1.25) the form:

$$\left.\begin{aligned}\varphi_{xx}\left(1-\frac{\varphi_x^2}{c^2}\right)+\varphi_{yy}\left(1-\frac{\varphi_y^2}{c^2}\right)+\varphi_{zz}\left(1-\frac{\varphi_z^2}{c^2}\right)-\\ -2\varphi_{xy}\frac{\varphi_x\varphi_y}{c^2}-2\varphi_{xz}\frac{\varphi_x\varphi_z}{c^2}-2\varphi_{yz}\frac{\varphi_y\varphi_z}{c^2}=0.\end{aligned}\right\}\tag{1.27}$$

The Eq. (1.27) is non-linear; it is, however, quasi-linear since the highest order derivatives (2nd order) occur linearly. It reduces to LAPLACE's equation in the limit case $c=\infty$ which can, therefore, be considered as the case of an incompressible fluid. The deviation from LAPLACE's equation will obviously be small if the ratio

$$M^2=\frac{q^2}{c^2}=\frac{\varphi_x^2+\varphi_y^2+\varphi_z^2}{c^2}\tag{1.28}$$

is small.

The quantity $M=q/c$ is called the local Mach number of the flow at the point x, y, z considered. It is a well determined function of the local speed q since c depends on q only. Its importance for the general theory is already obvious in the form (1.27) of the fundamental partial differential equation and will become more so in the further developments.

2. The case of the ideal gas. The most important equation of state considered in gas dynamics is that of an ideal adiabatic gas. With a proper choice of units we have POISSON's pressure density relation:

$$p=\varrho^\gamma,\tag{2.1}$$

where γ is the ratio of the specific heats of the gas for constant pressure and for constant volume:

$$\gamma=\frac{c_p}{c_v}.\tag{2.1'}$$

In the case of air the value $\gamma=1.4$ is theoretically and experimentally satisfactory.

The enthalpy of a fluid with the pressure density relation (2.1) is by Eq. (1.11'')

$$H(\varrho)=\frac{\gamma}{\gamma-1}\varrho^{\gamma-1},\tag{2.2}$$

and the Bernoulli equation takes the form

$$q^2+\frac{2\gamma}{\gamma-1}\varrho^{\gamma-1}=\text{const}.\tag{2.3}$$

The significance of the right hand constant is obvious; it represents the maximum velocity of the fluid corresponding to the value $\varrho=0$ and is attained when the fluid flows into vacuum. This constant is called the escape velocity and is denoted by $q_{\max}$. We can derive from Eqs. (2.3) and (2.1) the formulas

$$\frac{\gamma-1}{2\gamma}\left[q_{\max}^2-q^2\right]=\varrho^{\gamma-1}\tag{2.4}$$

and

$$q^2=q_{\max}^2-\frac{2\gamma}{\gamma-1}p^{\frac{\gamma-1}{\gamma}}.\tag{2.4'}$$

We normalize the units in such a way that to the state of rest, i.e. to $q=0$, correspond the values $\varrho=1$, $p=1$. We then derive from Eq. (2.4')

$$q_{\max}=\sqrt{\frac{2\gamma}{\gamma-1}}\tag{2.5}$$

and can rewrite Eq. (2.4') in the form

$$q^2 = q_{\max}^2 \left(1 - p^{\frac{\gamma-1}{\gamma}}\right). \tag{2.4''}$$

This is the formula of de St. Venant and Wantzel [3].

We obtain from Eq. (2.1) the formulas for the local speed of sound

$$c^2 = p'(\varrho) = \gamma \varrho^{\gamma-1} = \gamma \frac{p}{\varrho} = \gamma p^{\frac{\gamma-1}{\gamma}}. \tag{2.6}$$

In view of Eqs. (2.5) and (2.4'') we have the alternative forms

$$c^2 = \frac{\gamma-1}{2}(q_{\max}^2 - q^2) = \gamma - \frac{\gamma-1}{2} q^2. \tag{2.7}$$

c is thus expressed explicitly in terms of the local speed q. We observe that $q = q_{\max}$ implies $c = 0$ and that the maximum value of c

$$c_0 = \sqrt{\gamma} \tag{2.8}$$

is attained for the state of rest.

The speed q which coincides with its corresponding velocity of sound is called the critical speed q_{crit}. From (2.7) we compute

$$q_{\text{crit}} = \sqrt{\frac{2\gamma}{\gamma+1}} = \sqrt{\frac{2}{\gamma+1}}\, c_0. \tag{2.9}$$

To q_{crit} corresponds the Mach number $M = 1$; for $q < q_{\text{crit}}$ we have $M(q) < 1$ and for $q > q_{\text{crit}}$ holds $M(q) > 1$. A flow is called subsonic at a place where its local Mach number is less than 1 and supersonic where it is larger than 1. We have therefore the criterion

$$q < \sqrt{\frac{2\gamma}{\gamma+1}}, \text{ subsonic}; \qquad q > \sqrt{\frac{2\gamma}{\gamma+1}}, \text{ supersonic flow}. \tag{2.10}$$

It is of interest to point out a mathematical accident which is of considerable importance in gas dynamics. We can apply the binomial theorem and derive from Eq. (2.4'') the series development for the pressure in terms of the speed variable q:

$$p = \left[1 - \frac{\gamma-1}{2\gamma} q^2\right]^{\frac{\gamma}{\gamma-1}} = 1 - \frac{1}{2} q^2 + \frac{1}{8\gamma} q^4 + \cdots. \tag{2.11}$$

In the case of an incompressible fluid of constant density 1 and the pressure 1 at rest, we have the Bernoulli formula

$$p = 1 - \frac{1}{2} q^2. \tag{2.12}$$

It happens that the two series developments coincide in their first two terms. If we treat an ideal gas as incompressible, we commit an error in all pressure effects which is only of the orde $c_0^{-2} q^4$ and relatively small for smaller Mach numbers. This is the reason why the theory of incompressible fluid flows is quite satisfactory for gas dynamics even up to the Mach number 0.5.

From Eqs. (2.7) and (2.5), we get the formula for the Mach number

$$M^2 = \frac{1}{\gamma} \frac{q^2}{1 - \frac{q^2}{q_{\max}^2}}, \qquad q^2 = \frac{\gamma M^2}{1 + \frac{\gamma-1}{2} M^2} \tag{2.13}$$

and from Eq. (2.4) we derive

$$\varrho^{\gamma-1} = \left[1 + \frac{\gamma-1}{2} M^2\right]^{-1}. \tag{2.13'}$$

Let us consider a steady flow of a compressible medium. All stream lines through a fixed small circle form a narrow tube, called a stream tube. In the flow regime, no fluid passes through the walls of a stream tube. Consider next the stream line through the center of the original circle and denote by $P(s)$ a variable point on it; the parameter s may be chosen as the arc length along this curve. Finally, let $\pi(s)$ be the plane through $P(s)$ which is orthogonal to the stream line.

The stream tube will cut off from $\pi(s)$ a cross section of area $A(s)$. We assume the tube so narrow that the variation of density and speed over each cross section may be neglected and we denote the corresponding values of density and speed by $\varrho(s)$ and $q(s)$. We may then formulate the law of conservation of matter as

$$A(s)\,\varrho(s)\,q(s) = \text{const}. \tag{2.14}$$

By logarithmic differentiation of Eq. (2.14) we obtain

$$\frac{dA}{A} + \frac{d\varrho}{\varrho} + \frac{dq}{q} = 0. \tag{2.15}$$

On the other hand, we may express $d\varrho$ and dq in terms of dp by use of Eqs. (2.1) and (2.3); we have

$$\frac{dp}{p} = \gamma\frac{d\varrho}{\varrho}, \qquad q\,dq + \frac{1}{\varrho}\,dp = 0 \tag{2.16}$$

and, hence, using Eq. (2.6), we may bring Eq. (2.15) into the form

$$\frac{dA}{A} = \frac{dp}{\varrho q^2}\left(1 - \frac{\varrho q^2}{\gamma p}\right) = \frac{dp}{\varrho q^2}\left(1 - \frac{q^2}{c^2}\right). \tag{2.17}$$

It is easily seen that in the case of an incompressible fluid we have instead of Eq. (2.17)

$$\frac{dA}{A} = \frac{dp}{\varrho q^2}. \tag{2.17'}$$

The significance of the Mach number $M = q/c$ in the comparison between compressible and incompressible fluid flow is obvious.

From Eqs. (2.16) and (2.17) we find

$$\frac{dA}{A} = -\frac{dq}{q}\left(1 - \frac{q^2}{c^2}\right). \tag{2.18}$$

The cross section of a stream tube decreases with increasing speed until the sonic velocity is attained. At this moment, A attains its minimum and increases thereafter with increasing supersonic speed. This shows the basic difference in flow geometry and dynamics between the subsonic and supersonic flow regimes. The preceding considerations play an important role in turbine theory and in the theory of Laval nozzles. One purpose of such a nozzle is to create of gas flow of high supersonic speed. In order to surpass the sonic velocity in a flow through a Laval nozzle, the latter must narrow down until sonic velocity is attained and widen again after this point in order to make possible a further increase in the speed of flow.

3. Some explicit solutions of the fundamental equations. In finding particular solutions for the differential equations (1.27) of the velocity potential, it is often useful to bear in mind its genesis from the continuity equation. Let us ask, for example, for a solution $\varphi(r)$ which depends only on the distance r from a fixed point O, say the origin. Since $\boldsymbol{q} = \nabla\varphi$, the corresponding flow will be radial; that is, its stream lines will be radii from O and its speed $q(r)$ will depend on r only.

The conservation of matter will be expressed by the formula

$$4\pi r^2 \varrho q = \text{const} \tag{3.1}$$

which guarantees that the same amount of matter enters per unit of time through the inner wall of every concentric spherical shell as leaves through the outer wall. This result could also have been obtained by integration of the equation of continuity.

From Eqs. (3.1), (2.13) and (2.13′) we compute easily

$$r^2 = C \cdot \frac{1}{M}\left[1 + \frac{\gamma-1}{2} M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}. \tag{3.2}$$

We observe now that the right hand term in Eq. (3.2) cannot decrease indefinitely. It has a minimum for $M = 1$, namely

$$r_{\min} = C \cdot \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}. \tag{3.2′}$$

Thus, a radially symmetric solution of Eq. (1.27) is only possible outside of some critical sphere. While in the theory of an incompressible fluid we have point sources and sinks giving rise to radial flows, in the theory of compressible fluids we find radial sources and sinks possessing a spherical nucleus inside of which the mathematical solution breaks down and where the physical idealization become inapplicable. Here for the first time we encounter the phenomenon of a limit surface beyond which the mathematical theory of the physical assumptions cannot be continued; we shall meet this situation later in a more general context; see Sect. 33.

We observe that the acceleration of a radial flow is given by

$$a = q \cdot \frac{dq}{dr} = q \cdot \frac{dq}{dM} \cdot \frac{dM}{dr}. \tag{3.3}$$

As long as $q < q_{\max}$ we have clearly $dq/dM > 0$. On the other hand, it follows from Eq. (3.2) that $dr/dM = 0$ for $M = 1$. Hence, if we approach the critical radius $r_{\min}$ the acceleration in the flow approaches infinity.

Flows of radial nature occur in conical pipes and are well-described by formula (3.2) over the range of validity of this solution.

In precisely the same way we could solve the problem of a plane radial flow. Using the law of conservation of matter, we would find

$$r = r_{\min} \frac{1}{M}\left[\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1} M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}. \tag{3.4}$$

We would encounter a limit circle for the flow of radius $r_{\min}$ and corresponding to the Mach number $M = 1$.

Another explicit solution of the problem of a steady irrotational and homentropic flow is obtained when we consider a plane flow whose stream lines are concentric circles around a point O, say the origin, and where the speed $q(r)$

depends only on the distance r from O. We call such a flow a twodimensional vortex flow around O.

By the geometric nature of the flow the conservation of matter is automatically fulfilled and we have now only to guarantee the irrotationality of the motion. We observe that by Eq. (1.13) the circulation over a circle of radius r is given by

$$Z = 2\pi r q(r). \tag{3.5}$$

If the vorticity ζ is identically zero in every concentric ring around 0, the application of Eq. (1.15) to the ring domain shows immediately that Z is a constant. Thus Eq. (3.5) represents $q(r)$ as a function of r; by use of Eq. (2.13) we have

$$r^2 = C\,\frac{2+(\gamma-1)\,M^2}{M^2}. \tag{3.6}$$

We see again that the radius r cannot decrease indefinitely but has the minimum

$$r_{\min}^2 = (\gamma-1)\,C \tag{3.7}$$

which corresponds to the Mach number infinity, that is, to the escape velocity $q_{\max}$.

If we write

$$r^2 = r_{\min}^2\left[1+\frac{2}{\gamma-1}\cdot\frac{1}{M^2}\right] \tag{3.6'}$$

we see clearly that the two-dimensional vortex flow is subsonic for

$$r > r_{\min}\sqrt{\frac{\gamma+1}{\gamma-1}} \tag{3.8}$$

and supersonic for

$$r < r_{\min}\sqrt{\frac{\gamma+1}{\gamma-1}}. \tag{3.8}$$

Since by Eq. (2.4) $q = q_{\max}$ implies the value $\varrho = 0$, we see that the two-dimensional vortex flow has a vacuum core of radius $r_{\min}$.

The two plane flows considered so far are both radially symmetric; that is, in both the velocity vector $q(r)$ depends only upon the distance from the origin. It is equally easy to determine the most general radially symmetric flow [4] to [6] by applying at the same time the conditions of irrotationality and of conservation of matter. For this purpose decompose the velocity q into the radial component $a(r)$ and the angular component $b(r)$. The law of conservation of matter requires

$$2\pi r\varrho\, a(r) = C \tag{3.9}$$

while the condition of irrotationality or constant circulation affects only $b(r)$:

$$2\pi r\, b(r) = Z. \tag{3.10}$$

Since $q^2 = a^2 + b^2$, we derive from Eqs. (2.13), (2.13'), (3.9) and (3.10):

$$r^2 = \frac{C^2}{4\pi^2\gamma}\,\frac{1}{M^2}\left[1+\frac{\gamma-1}{2}M^2\right]^{\frac{\gamma+1}{(\gamma-1)}} + \frac{Z^2}{4\pi^2\gamma}\cdot\frac{1}{M^2}\left[1+\frac{\gamma-1}{2}M^2\right]. \tag{3.11}$$

It is again apparent that r has a minimum value $r_{\min}$, which can be derived from Eq. (3.11) by differentiation. We obtain a more intuitive characterization by using the identity

$$4\pi^2 r^2 q^2 = \frac{C^2}{\varrho^2} + Z^2 \tag{3.12}$$

and differentiating it with respect to q^2; using Eq. (1.26) we find

$$4\pi^2 q^2 \frac{dr^2}{dq^2} + 4\pi^2 r^2 = + \frac{C^2}{\varrho^2 c^2}. \tag{3.12'}$$

For the minimum value of r, the first term in Eq. (3.12′) must vanish and we find

$$4\pi^2 r_{\min}^2 \varrho^2 c^2 = C^2. \tag{3.12''}$$

Comparing Eqs. (3.9) with (3.12″) we see that the minimum radius is attained when the radial speed equals the speed of sound, that is, for sonic radial speed.

There are two values of M^2 possible for a given value of r^2; the one M-value corresponds to subsonic radial speed and the other to supersonic values of $a(r)$. Two different flow patterns, therefore, are possible according to the branch $M(r)$ chosen in the domain $r > r_{\min}$.

In order to calculate the streamlines of the flow obtained, we start with the differential equation

$$\frac{d\vartheta}{dr} = \frac{b(r)}{r\,a(r)} = \frac{Z}{C} \cdot \frac{\varrho}{r}. \tag{3.13}$$

Since ϱ is a simple function of M and the relation between r and M is given by Eq. (3.11) we can always find the function $\vartheta(r)$ describing the streamlines in polar coordinates. The calculation becomes quite elementary when we choose $\gamma = 1.4$, that is, $\frac{7}{5}$ exactly, and introduce the variable

$$l = (1 + \tfrac{1}{5} M^2)^{-\frac{1}{2}}. \tag{3.14}$$

We find from Eqs. (2.13′) and (3.11)

$$\varrho = l^5, \qquad r^2 = \frac{\alpha^2 l^{-10} + \beta^2}{1 - l^2} \tag{3.15}$$

and from Eq. (3.13) we can compute $\vartheta(l)$ in terms of elementary functions. We find

$$\vartheta = \frac{\beta}{\alpha} \left[\log \sqrt{\frac{1+l}{1-l}} - l - \frac{1}{3} l^3 - \frac{1}{5} l^5\right] - \arctan \frac{\beta}{\alpha} l^5 + \text{const}. \tag{3.15'}$$

All streamlines are obtained from a representative one by turning by a fixed angle since the general equation is $\vartheta = \vartheta_0(r) + \text{const}$.

The flow described by Eqs. (3.15), (3.15′) has streamlines in forms of spirals starting from the limit circle $r = r_{\min}$. For a detailed description, see RINGLEB [7]. See also [8], [9].

4. Plane and axially symmetric flows. A useful simplification of the partial differential equation for the velocity potential φ is possible in the case of a two-dimensional flow. Here φ depends only on the two variables, x, y, and the Eq. (1.22) takes the form

$$\frac{\partial}{\partial x}(\varrho \varphi_x) + \frac{\partial}{\partial y}(\varrho \varphi_y) = 0. \tag{4.1}$$

This can be interpreted as the integrability condition for a new function $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = -\varrho \frac{\partial \varphi}{\partial y} = -\varrho v, \qquad \frac{\partial \psi}{\partial y} = \varrho \frac{\partial \varphi}{\partial x} = \varrho u. \tag{4.2}$$

The hydrodynamical derivative of ψ is

$$\frac{D\psi}{Dt} = \psi_x u + \psi_y v = \varrho(-\varphi_x \varphi_y + \varphi_x \varphi_y) = 0; \tag{4.3}$$

that is, $\psi(x, y)$ is constant along each streamline of the flow. $\psi(x, y)$ is called the *stream function* of the flow.

If C is an arbitrary curve in the (x, y)-plane connecting the points x_0, y_0 and x_1, y_1 the integral

$$Q = \int_{x_0, y_0}^{x_1, y_1} \varrho (u\,dy - v\,dx) \tag{4.4}$$

extended over C represents the amount of matter carried per unit of time across this curve. By virtue of Eq. (4.2) we can write

$$Q = \psi(x_1, y_1) - \psi(x_0, y_0). \tag{4.4'}$$

Thus, the difference of the stream function values at two points represents the flux of matter through any curve connecting them. This is another intuitive interpretation of the stream function which reveals its close connection with the continuity equation.

If we eliminate ψ from the system of first order differential equations (4.2) we obtain again Eq. (4.1), and by virtue of Eq. (1.26) follows

$$\left(1 - \frac{\varphi_x^2}{c^2}\right)\varphi_{xx} + \left(1 - \frac{\varphi_y^2}{c^2}\right)\varphi_{yy} - 2\frac{\varphi_x \varphi_y}{c^2}\varphi_{xy} = 0, \tag{4.5}$$

which is a particular case of Eq. (1.27). In order to eliminate φ from the system (4.2) we observe the identity

$$(\psi_x^2 + \psi_y^2) = \varrho^2 q^2. \tag{4.6}$$

We differentiate this with respect to x and y and use the differential relation (1.26). We obtain

$$\psi_x \psi_{xx} + \psi_y \psi_{xy} = \varrho_x \varrho (q^2 - c^2), \qquad \psi_x \psi_{xy} + \psi_y \psi_{yy} = \varrho_y \varrho (q^2 - c^2). \tag{4.7}$$

On the other hand, we obtain from Eq. (4.2) directly

$$\varrho(\psi_{xx} + \psi_{yy}) - (\varrho_x \psi_x + \varrho_y \psi_y) = 0. \tag{4.8}$$

Combining Eqs. (4.7) with (4.8) we are led finally to the differential equation for the stream function

$$\psi_{xx}\left(1 - \frac{\psi_y^2}{c^2 \varrho^2}\right) + \psi_{yy}\left(1 - \frac{\psi_x^2}{c^2 \varrho^2}\right) + 2\psi_{xy}\frac{\psi_x \psi_y}{c^2 \varrho^2} = 0. \tag{4.9}$$

In this equation $c\varrho$ is to be considered as a function of $\psi_x^2 + \psi_y^2$ computed by means of Eq. (4.6) and the known relations between c, ϱ and q.

The stream function is particularly useful in the study of flows with prescribed fixed boundaries. The presence of rigid walls subjects the velocity potential to the boundary condition $\partial\varphi/\partial n = 0$, which guarantees that the flow does not cross the boundary; the stream function, on the other hand, satisfies the boundary condition $\psi = \text{const}$, which means that the boundary curve must be a streamline. In general, it is much easier to find the solution of a partial differential equation with prescribed boundary values (DIRICHLET'S problem) than to find a solution with specified normal derivative on the boundary (NEUMANN'S problem). This fact explains the importance of the stream function in the theory of two-dimensional flows around given profiles.

The significance of the system (4.2) is well illustrated by the limit case of an incompressible fluid with density 1. In this case, the system reduces to the classical Cauchy-Riemann equations

$$\varphi_x = \psi_y, \qquad \varphi_y = -\psi_x \tag{4.10}$$