CLASSICAL MECHANICS

SECOND EDITION

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Columbia University

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PREFACE TO THE SECOND EDITION

The prospect of a second edition of *Classical Mechanics*, almost thirty years after initial publication, has given rise to two nearly contradictory sets of reactions. On the one hand it is claimed that the adjective "classical" implies the field is complete, closed, far outside the mainstream of physics research. Further, the first edition has been paid the compliment of continuous use as a text since it first appeared. Why then the need for a second edition? The contrary reaction has been that a second edition is long overdue. More important than changes in the subject matter (which have been considerable) has been the revolution in the attitude towards classical mechanics in relation to other areas of science and technology. When it appeared, the first edition was part of a movement breaking with older ways of teaching physics. But what were bold new ventures in 1950 are the commonplaces of today, exhibiting to the present generation a slightly musty and old-fashioned air. Radical changes need to be made in the presentation of classical mechanics.

In preparing this second edition I have attempted to steer a course somewhere between these two attitudes. I have tried to retain, as much as possible, the advantages of the first edition (as I perceive them) while taking some account of the developments in the subject itself, its position in the curriculum, and its applications to other fields. What has emerged is a thorough-going revision of the first edition. Hardly a page of the text has been left untouched. The changes have been of various kinds:

Errors (some egregious) that I have caught, or which have been pointed out to me, have of course been corrected. It is hoped that not too many new ones have been introduced in the revised material.

The chapter on small oscillations has been moved from its former position as the penultimate chapter and placed immediately after Chapter 5 on rigid body motion. This location seems more appropriate to the usual way mechanics courses are now being given. Some material relating to the Hamiltonian formulation has therefore had to be removed and inserted later in (the present) Chapter 8.

A new chapter on perturbation theory has been added (Chapter 11). The last chapter, on continuous systems and fields, has been greatly expanded, in keeping with the implicit promise made in the Preface to the first edition.

New sections have been added throughout the book, ranging from one in Chapter 3 on Bertrand's theorem for the central-force potentials giving rise to closed orbits, to the final section of Chapter 12 on Noether's theorem. For the most part these sections contain completely new material.

In various sections arguments and proofs have been replaced by new ones that seem simpler and more understandable, e.g., the proof of Euler's theorem in Chapter 4. Occasionally, a line of reasoning presented in the first edition has been supplemented by a different way of looking at the problem. The most important example is the introduction of the symplectic approach to canonical transformations, in parallel with the older technique of generating functions. Again, while the original convention for the Euler angles has been retained, alternate conventions, including the one common in quantum mechanics, are mentioned and detailed formulas are given in an appendix.

As part of the fruits of long experience in teaching courses based on the book, the body of exercises at the end of each chapter has been expanded by more than a factor of two and a half. The bibliography has undergone similar expansion, reflecting the appearance of many valuable texts and monographs in the years since the first edition. In deference to—but not in agreement with—the present neglect of foreign languages in graduate education in the United States, references to foreign-language books have been kept down to a minimum.

The choices of topics retained and of the new material added reflect to some degree my personal opinions and interests, and the reader might prefer a different selection. While it would require too much space (and be too boring) to discuss the motivating reasons relative to each topic, comment should be made on some general principles governing my decisions. The question of the choice of mathematical techniques to be employed is a vexing one. The first edition attempted to act as a vehicle for introducing mathematical tools of wide usefulness that might be unfamiliar to the student. In the present edition the attitude is more one of caution. It is much more likely now than it was 30 years ago that the student will come to mechanics with a thorough background in matrix manipulation. The section on matrix properties in Chapter 4 has nonetheless been retained, and even expanded, so as to provide a convenient reference of needed formulas and techniques. The cognoscenti can, if they wish, simply skip the section. On the other hand, very little in the way of newer mathematical tools has been introduced. Elementary properties of group theory are given scattered mention throughout the book. Brief attention is paid in Chapters 6 and 7 to the manipulation of tensors in non-Euclidean spaces. Otherwise, the mathematical level in this edition is pretty much the same as in the first. It is more than adequate for the physics content of the book, and alternate means exist in the curriculum for acquiring the mathematics needed in other branches of physics. In particular the "new mathematics" of theoretical physics has been deliberately excluded. No mention is made of manifolds or diffeomorphisms, of tangent fibre bundles or invariant torii. There are certain highly specialized areas of classical mechanics where the powerful tools of global analysis and differential topology are useful, probably essential. However, it is not clear to me that they contribute to the understanding of the physics of classical mechanics at the level sought in this edition. To introduce these mathematical concepts, and their applications, would swell the book beyond bursting, and serve, probably, only to obscure the physics. Theoretical physics, current trends to the contrary, is not merely mathematics.

In line with this attitude, the complex Minkowski space has been retained for most of the discussion of special relativity in order to simplify the mathematics. The bases for this decision (which it is realized goes against the present fashion) are given in detail on pages 292–293.

It is certainly true that classical mechanics today is far from being a closed subject. The last three decades have seen an efflorescence of new developments in classical mechanics, the tackling of new problems, and the application of the techniques of classical mechanics to far-flung reaches of physics and chemistry. It would clearly not be possible to include discussions of all of these developments here. The reasons are varied. Space limitations are obviously important. Also, popular fads of current research often prove ephemeral and have a short lifetime. And some applications require too extensive a background in other fields, such as solid-state physics or physical chemistry. The selection made here represents something of a personal compromise. Applications that allow simple descriptions and provide new insights are included in some detail. Others are only briefly mentioned, with enough references to enable the student to follow up his awakened curiosity. In some instances I have tried to describe the current state of research in a field almost entirely in words, without mathematics, to provide the student with an overall view to guide further exploration. One area omitted deserves special mention—nonlinear oscillation and associated stability questions. The importance of the field is unquestioned, but it was felt that an adequate treatment deserves a book to itself.

With all the restrictions and careful selection, the book has grown to a size probably too large to be covered in a single course. A number of sections have been written so that they may be omitted without affecting later developments and have been so marked. It was felt however that there was little need to mark special "tracks" through the book. Individual instructors, familiar with their own special needs, are better equipped to pick and choose what they feel should be included in the courses they give.

I am grateful to many individuals who have contributed to my education in classical mechanics over the past thirty years. To my colleagues Professors Frank L. DiMaggio, Richard W. Longman, and Dean Peter W. Likins I am indebted for many valuable comments and discussions. My thanks go to Sir Edward Bullard for correcting a serious error in the first edition, especially for the gentle and gracious way he did so. Professor Boris Garfinkel of Yale University very kindly read and commented on several of the chapters and did his best to initiate me into the mysteries of celestial mechanics. Over the years I have been the grateful recipient of valuable corrections and suggestions from many friends and strangers, among whom particular mention should be made of Drs. Eric Ericsen (of Oslo University), K. Kalikstein, J. Neuberger, A. Radkowsky, and Mr. W. S. Pajes. Their contributions have certainly enriched the book, but of course I alone am responsible for errors and misinterpretations. I should like to add a collective acknowledgment and thanks to the authors of papers on classical mechanics that have appeared during the last three decades in the *American Journal of Physics,* whose pages I hope I have perused with profit.

The staff at Addison-Wesley have been uniformly helpful and encouraging. I want especially to thank Mrs. Laura R. Finney for her patience with what must have seemed a never-ending process, and Mrs. Marion Howe for her gentle but persistent cooperation in the fight to achieve an acceptable printed page.

To my father, Harry Goldstein ז"ל, I owe more than words can describe for his lifelong devotion and guidance. But I wish at least now to do what he would not permit in his lifetime—to acknowledge the assistance of his incisive criticism and careful editing in the preparation of the first edition. I can only hope that the present edition still reflects something of his insistence on lucid and concise writing.

I wish to dedicate this edition to those I treasure above all else on this earth, and who have given meaning to my life—to my wife, Channa, and our children, Penina Perl, Aaron Meir, and Shoshanna.

And above all I want to register the thanks and acknowledgment of my heart, in the words of Daniel (2:23):

לך אלה אבהתי מהודא ומשבח אנה
די חכמתא וגבורתא יהבת לי

Kew Gardens Hills, New York HERBERT GOLDSTEIN
January 1980

PREFACE TO THE FIRST EDITION

An advanced course in classical mechanics has long been a time-honored part of the graduate physics curriculum. The present-day function of such a course, however, might well be questioned. It introduces no new physical concepts to the graduate student. It does not lead him directly into current physics research. Nor does it aid him, to any appreciable extent, in solving the practical mechanics problems he encounters in the laboratory.

Despite this arraignment, classical mechanics remains an indispensable part of the physicist's education. It has a twofold role in preparing the student for the study of modern physics. First, classical mechanics, in one or another of its advanced formulations, serves as the springboard for the various branches of modern physics. Thus, the technique of action-angle variables is needed for the older quantum mechanics, the Hamilton-Jacobi equation and the principle of least action provide the transition to wave mechanics, while Poisson brackets and canonical transformations are invaluable in formulating the newer quantum mechanics. Secondly, classical mechanics affords the student an opportunity to master many of the mathematical techniques necessary for quantum mechanics while still working in terms of the familiar concepts of classical physics.

Of course, with these objectives in mind, the traditional treatment of the subject, which was in large measure fixed some fifty years ago, is no longer adequate. The present book is an attempt at an exposition of classical mechanics which does fulfill the new requirements. Those formulations which are of importance for modern physics have received emphasis, and mathematical techniques usually associated with quantum mechanics have been introduced wherever they result in increased elegance and compactness. For example, the discussion of central force motion has been broadened to include the kinematics of scattering and the classical solution of scattering problems. Considerable space has been devoted to canonical transformations, Poisson bracket formulations, Hamilton-Jacobi theory, and action-angle variables. An introduction has been provided to the variational principle formulation of continuous systems and fields. As an illustration of the application of new mathematical techniques, rigid body rotations are treated from the standpoint of matrix transformations. The familiar Euler's theorem on the motion of a rigid body can then be presented in terms of the eigenvalue problem for an orthogonal matrix. As a consequence, such diverse topics as the inertia tensor, Lorentz transformations in Minkowski space, and resonant frequencies of small oscillations become capable of a unified mathematical treatment. Also, by this technique it becomes possible to include at an early stage the difficult concepts of reflection operations and pseudotensor quantities, so important in modern quantum mechanics. A further advantage of matrix methods is that "spinors" can be introduced in connection with the properties of Cayley-Klein parameters.

Several additional departures have been unhesitatingly made. All too often, special relativity receives no connected development except as part of a highly specialized course which also covers general relativity. However, its vital importance in modern physics requires that the student be exposed to special relativity at an early stage in his education. Accordingly, Chapter 6 has been devoted to the subject. Another innovation has been the inclusion of velocity-dependent forces. Historically, classical mechanics developed with the emphasis on static forces dependent on position only, such as gravitational forces. On the other hand, the velocity-dependent electromagnetic force is constantly encountered in modern physics. To enable the student to handle such forces as early as possible, velocity-dependent potentials have been included in the structure of mechanics from the outset, and have been consistently developed throughout the text.

Still another new element has been the treatment of the mechanics of continuous systems and fields in Chapter 11, and some comment on the choice of material is in order. Strictly interpreted, the subject could include all of elasticity, hydrodynamics, and acoustics, but these topics lie outside the prescribed scope of the book, and adequate treatises have been written for most of them. In contrast, no connected account is available on the classical foundations of the variational principle formulation of continuous systems, despite its growing importance in the field theory of elementary particles. The theory of fields can be carried to considerable length and complexity before it is necessary to introduce quantization. For example, it is perfectly feasible to discuss the stress-energy tensor, microscopic equations of continuity, momentum space representations, etc., entirely within the domain of classical physics. It was felt, however, that an adequate discussion of these subjects would require a sophistication beyond what could naturally be expected of the student. Hence it was decided, for this edition at least, to limit Chapter 11 to an elementary description of the Lagrangian and Hamiltonian formulation of fields.

The course for which this text is designed normally carries with it a prerequisite of an intermediate course in mechanics. For both the inadequately prepared graduate student (an all too frequent occurrence) and the ambitious senior who desires to omit the intermediate step, an effort was made to keep the book self-contained. Much of Chapters 1 and 3 is therefore devoted to material usually covered in the preliminary courses.

With few exceptions, no more mathematical background is required of the student than the customary undergraduate courses in advanced calculus and vector analysis. Hence considerable space is given to developing the more complicated mathematical tools as they are needed. An elementary acquaintance with Maxwell's equations and their simpler consequences is necessary for understanding the sections on electromagnetic forces. Most entering graduate students have had at least one term's exposure to modern physics, and frequent advantage has been taken of this circumstance to indicate briefly the relation between a classical development and its quantum continuation.

A large store of exercises is available in the literature on mechanics, easily accessible to all, and there consequently seemed little point to reproducing an extensive collection of such problems. The exercises appended to each chapter therefore have been limited, in the main, to those which serve as extensions of the text, illustrating some particular point or proving variant theorems. Pedantic museum pieces have been studiously avoided.

The question of notation is always a vexing one. It is impossible to achieve a completely consistent and unambiguous system of notation that is not at the same time impracticable and cumbersome. The customary convention has been followed of indicating vectors by bold face Roman letters. In addition, matrix quantities of whatever rank, and tensors other than vectors, are designated by bold face sans serif characters, thus: **A**. An index of symbols is appended at the end of the book, listing the initial appearance of each meaning of the important symbols. Minor characters, appearing only once, are not included.

References have been listed at the end of each chapter, for elaboration of the material discussed or for treatment of points not touched on. The evaluations accompanying these references are purely personal, of course, but it was felt necessary to provide the student with some guide to the bewildering maze of literature on mechanics. These references, along with many more, are also listed at the end of the book. The list is not intended to be in any way complete, many of the older books being deliberately omitted. By and large, the list contains the references used in writing this book, and must therefore serve also as an acknowledgement of my debt to these sources.

The present text has evolved from a course of lectures on classical mechanics that I gave at Harvard University, and I am grateful to Professor J. H. Van Vleck, then Chairman of the Physics Department, for many personal and official encouragements. To Professor J. Schwinger, and other colleagues I am indebted for many valuable suggestions. I also wish to record my deep gratitude to the students in my courses, whose favorable reaction and active interest provided the continuing impetus for this work.

תושלב"ע

Cambridge, Mass.
March 1950

HERBERT GOLDSTEIN

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CHAPTER 1
Survey of the Elementary Principles

The motion of material bodies formed the subject of some of the earliest researches pursued by the pioneers of physics. From their efforts there has evolved a vast field known as analytical mechanics or dynamics, or simply, mechanics. In the present century the term "classical mechanics" has come into wide use to denote this branch of physics in contradistinction to the newer physical theories, especially quantum mechanics. We shall follow this usage, interpreting the name to include the type of mechanics arising out of the special theory of relativity. It is the purpose of this book to develop the structure of classical mechanics and to outline some of its applications of present-day interest in pure physics.

Basic to any presentation of mechanics are a number of fundamental physical concepts, such as space, time, simultaneity, mass, and force. In discussing the special theory of relativity the notions of simultaneity and of time and length scales will be examined briefly. For the most part, however, these concepts will not be analyzed critically here; rather, they will be assumed as undefined terms whose meanings are familiar to the reader.

1–1 MECHANICS OF A PARTICLE

Let $\mathbf{r}$ be the radius vector of a particle from some given origin and $\mathbf{v}$ its vector velocity:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}. \tag{1–1}$$

The *linear momentum* $\mathbf{p}$ of the particle is defined as the product of the particle mass and its velocity:

$$\mathbf{p} = m\mathbf{v}. \tag{1–2}$$

In consequence of interactions with external objects and fields the particle may experience forces of various types, e.g., gravitational or electrodynamic; the vector sum of these forces exerted on the particle is the total force $\mathbf{F}$. The mechanics of the particle is contained in *Newton's Second Law of Motion*, which states that there exist frames of reference in which the motion of the particle is described by the differential equation

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \tag{1-3}$$

or

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}). \tag{1-4}$$

In most instances the mass of the particle is constant and Eq. (1–3) reduces to

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}, \tag{1-5}$$

where **a** is the vector acceleration of the particle defined by

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}. \tag{1-6}$$

The equation of motion is thus a differential equation of second order, assuming **F** does not depend on higher order derivatives.

A reference frame in which Eq. (1–3) is valid is called an *inertial* or *Galilean system*. Even within classical mechanics the notion of an inertial system is something of an idealization. In practice, however, it is usually feasible to set up a coordinate system that comes as close to the desired properties as may be required. For many purposes a reference frame fixed in the Earth (the "laboratory system") is a sufficient approximation to an inertial system, while for some astronomical purposes it may be necessary to construct an inertial system by reference to the most distant galaxies.

Many of the important conclusions of mechanics can be expressed in the form of conservation theorems, which indicate under what conditions various mechanical quantities are constant in time. Equation (1–1) directly furnishes the first of these, the

Conservation Theorem for the Linear Momentum of a Particle: If the total force, **F**, *is zero then* $\dot{\mathbf{p}} = 0$ *and the linear momentum*, **p**, *is conserved.*

The angular momentum of the particle about point *O*, denoted by **L**, is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{1-7}$$

where **r** is the radius vector from *O* to the particle. Notice that the order of the factors is important. We now define the *moment of force* or *torque* about *O* as

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}. \tag{1-8}$$