C^*-ALGEBRAS AND THEIR AUTOMORPHISM GROUPS

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GERT K. PEDERSEN

Mathematics Institute
University of Copenhagen
Copenhagen, Denmark

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Preface

The theory of C^*-algebras is the study of operators on Hilbert space with algebraic methods. The motivating example is the spectral theorem for a normal operator (which, in effect, is nothing but Gelfand transformation applied to the algebra generated by the operator). The applications of the theory range from group representations to model quantum field theory and quantum statistical mechanics.

Already the C^*-algebra theory has grown to a size where any comprehensive treatment would result in a series of volumes more suited as a source of references than as a textbook. The material presented here has been limited by the author's knowledge and prejudice to form a somewhat manageable version. Thus the aspects of the theory concerning partially ordered vector spaces are treated in great detail. Also, since C^*-algebra theory has benefited tremendously from impulses from mathematical physics, it seemed proper to give an account which would please the C^*-physicists. Therefore the problems connected with groups of automorphisms have received special attention in this treatise. In the converse direction, the theory of von Neumann algebras, often so dominantly exposed, has here been reduced to its proper place as ancilla C^*-algebrae.

At the end of each section a few remarks are inserted, with references to the bibliography. The intention is to give the reader a rough idea of the development of the subject. Such personal comments are bound to contain errors, and the author humbly asks forgiveness from the mathematicians who have undeservedly not been mentioned.

Many people were important for the completion of this book: Richard Kadison whose work has been a constant source of inspiration for me; Daniel Kastler who provided shelter and a two months raincurtain when the work was begun in 1974; colleagues who shouldered my teaching load while I was writing; and students at the University of Copenhagen who were exposed to the first wildly incorrect drafts. It is a pleasure to record my thanks to all of them.

Copenhagen
August 1978

Gert Kjærgård Pedersen

Contents

Chapter 1

Abstract C^*-algebras

1.1 Spectral theory

1.1.1. A *C^*-algebra* is a complex Banach algebra A with an involution, $*$, satisfying $\|x^*x\| = \|x\|^2$ for all x in A. Since $\|x^*x\| \leq \|x^*\|\,\|x\|$ we have $\|x\| \leq \|x^*\|$ for each x in A, whence $\|x\| = \|x^*\|$, so that the involution is isometric. An element x in A is *normal* if it commutes with its *adjoint* x^*; and it is *self-adjoint* if $x = x^*$. The self-adjoint part of a subset B of A is denoted by B_{sa}. For each x in A the elements $\frac{1}{2}(x + x^*)$ and $-\frac{1}{2}i(x - x^*)$ (the real resp. imaginary part of x) belong to A_{sa}. It follows that A_{sa} is a closed real subspace of A and that each element x in A has a unique decomposition $x = y + iz$ with y and z in A_{sa}.

1.1.2. In general a C^*-algebra need not have a unit. If however, the C^*-algebra A has a unit (denoted by 1_A, or just 1 when no confusion may arise) and $A \neq 0$, then $1_A^* = 1_A$ and $\|1_A\| = 1$.

If $1 \in A$ we say that an element u in A is *unitary* if $u^*u = uu^* = 1$. Note that each unitary is normal and has norm 1.

1.1.3. PROPOSITION. *For each C^*-algebra A there is a C^*-algebra $\tilde{A}$ with unit containing A as a closed ideal. If A has no unit then $\tilde{A}/A = \mathbf{C}$.*

Proof. Let π denote the left regular representation of A as operators on itself, i.e. $\pi(x)y = xy$ for all x and y in A. It is clear that π is a homomorphism and that $\|\pi(x)\| \leq \|x\|$. But since

$$\|x\|^2 = \|xx^*\| = \|\pi(x)x^*\| \leq \|\pi(x)\|\,\|x^*\|$$

we see that π is an isometry. Let 1 denote the identity operator on A and let $\tilde{A}$ be the algebra of operators on A of the form $\pi(x) + \alpha 1$ with x in A and α in $\mathbf{C}$. Since $\pi(A)$ is complete and $\tilde{A}/\pi(A) = \mathbf{C}$, $\tilde{A}$ is also complete. With the involution defined by $(\pi(x) + \alpha 1)^* = \pi(x^*) + \bar{\alpha} 1$, $\tilde{A}$ becomes a C^*-algebra since for each $\varepsilon > 0$ there is a y in A with $\|y\| = 1$ such that

$$\|\pi(x) + \alpha 1\|^2 \leq \varepsilon + \|(x+\alpha)y\|^2$$
$$= \varepsilon + \|y^*(x^* + \bar{\alpha})(x+\alpha)y\| \leq \varepsilon + \|(x^* + \bar{\alpha})(x+\alpha)y\|$$
$$\leq \varepsilon + \|(\pi(x) + \alpha 1)^*(\pi(x) + \alpha 1)\|.$$

1.1.4. For each x in a C^*-algebra A we define the *spectrum* of x in A (written $\mathrm{Sp}_A(x)$) as the set of complex numbers λ such that $\lambda 1 - x$ is not invertible in $\tilde{A}$. Note that $0 \in \mathrm{Sp}_A(x)$ whenever $A \neq \tilde{A}$. By straightforward computations it follows that if $\lambda \neq 0$, then $\lambda \notin \mathrm{Sp}_A(x)$ if and only if there is a y in A such that $xy = yx = \lambda^{-1}x + \lambda y$ (corresponding to the fact that $\lambda^{-1} - y = (\lambda 1 - x)^{-1}$ in $\tilde{A}$).

If $x \in A_{sa}$ and $v(x)$ is the spectral radius of x then by repeated use of the equality $\|x^2\| = \|x\|^2$ we obtain

$$v(x) = \mathrm{Lim}\,\|x^{2^n}\|^{2^{-n}} = \|x\|.$$

If x is just normal, then from the preceding,

$$v(x)^2 \leq \|x\|^2 = \|x^*x\| = \mathrm{Lim}\,\|(x^*x)^n\|^{n^{-1}}$$
$$\leq \mathrm{Lim}(\|(x^*)^n\|\,\|x^n\|)^{n^{-1}} = v(x)^2,$$

whence again $v(x) = \|x\|$.

1.1.5. LEMMA. *If* $x \in A_{sa}$ *then* $\mathrm{Sp}(x) \subset \boldsymbol{R}$. *If* $1 \in A$ *and u is unitary*, $\mathrm{Sp}(u)$ *is contained in the unit circle.*

Proof. If $\lambda \in \mathrm{Sp}(u)$ then $\lambda^{-1} \in \mathrm{Sp}(u^{-1})$. Since $u^{-1} = u^*$ we have $|\lambda| \leq 1$ and $|\lambda^{-1}| \leq 1$, whence $|\lambda| = 1$ which proves the second assertion in the lemma.

Take now x in A_{sa}. The power series $\Sigma(n!)^{-1}(ix)^n$ converges in $\tilde{A}$ to an element $\exp(ix)$ which is unitary (in $\tilde{A}$) since

$$\exp(ix)^* = \exp(-ix) = \exp(ix)^{-1}.$$

If $\lambda \in \mathrm{Sp}(x)$ then $\exp(i\lambda) \in \mathrm{Sp}(\exp(ix))$ by computation, whence $|\exp(i\lambda)| = 1$ by the first part of the proof. It follows that $\mathrm{Sp}(x) \subset \boldsymbol{R}$ as desired.

1.1.6. Let A be a commutative Banach algebra. The *spectrum* $\hat{A}$ of A is the set of non-zero homomorphisms of A onto $\boldsymbol{C}$. Each element in $\hat{A}$ belongs to the unit ball of the dual A^* of A, and since $\hat{A} \cup \{0\}$ is the weak* closed subset of A^* consisting of functionals t such that $t(xy) = t(x)t(y)$ for all x, y in A, we see that $\hat{A}$ is a locally compact Hausdorff space in the weak* topology. The *Gelfand transform* on A is the homomorphism $x \to \hat{x}$ of A into $C_0(\hat{A})$ given by $\hat{x}(t) = t(x)$ for all x in A and t in $\hat{A}$.

1.1.7. THEOREM. *If A is a commutative C^*-algebra then the Gelfand transform is a $*$-preserving isometry of A onto $C_0(\hat{A})$.*

Proof. If $t \in \hat{A}$ and $x \in A$ then $\ker t$ is a maximal ideal of A, whence $t(x) \in \mathrm{Sp}(x)$ (and conversely, if $\lambda \in \mathrm{Sp}(x)\backslash\{0\}$ then $\lambda = t(x)$ for some t in $\hat{A}$). If therefore $x = x^*$ then $t(x) \in \boldsymbol{R}$ by 1.1.5. It follows that $t(x^*) = \overline{t(x)}$ for each x in A which shows that the map $x \to \hat{x}$ is $*$-preserving (using complex conjugation of functions as involution in $C_0(\hat{A})$). Moreover, $\|\hat{x}\|$ is the spectral radius of x, whence $\|\hat{x}\| = \|x\|$ by 1.1.4 as each x in A is normal. Thus $x \to \hat{x}$ is a $*$-preserving isometry of A into $C_0(\hat{A})$ and since the set of functions $\{\hat{x} \mid x \in A\}$ separates points in $\hat{A}$ and does not all vanish at any point we conclude from the Stone–Weierstrass theorem that the image of A is all of $C_0(\hat{A})$.

1.1.8. PROPOSITION. *Let x be a normal element of a C^*-algebra A, and let B denote the smallest C^*-subalgebra of A containing x. Then $B = C_0(\mathrm{Sp}_A(x)\backslash\{0\})$ and $\mathrm{Sp}_A(x)\backslash\{0\} = \mathrm{Sp}_B(x)\backslash\{0\}$.*

Proof. Since B is a singly generated commutative C^*-algebra it follows that $\hat{B} = \mathrm{Sp}_B(x)\backslash\{0\}$, whence $B = C_0(\mathrm{Sp}_B(x)\backslash\{0\})$ by 1.1.7 (suppressing the isomorphism). If therefore $\lambda \in \mathrm{Sp}_B(x)\backslash\{0\}$ there is for each $\varepsilon > 0$ an element b in B with $\|b\| = 1$ such that $\|\lambda b - xb\| < \varepsilon$. This shows that $\lambda 1 - x$ is not invertible in $\tilde{A}$, thus $\lambda \in \mathrm{Sp}_A(x)$. It is immediate from 1.1.4 that if $\lambda \notin \mathrm{Sp}_B(x)$ and $\lambda \neq 0$ then $\lambda \notin \mathrm{Sp}_A(x)$, and the proposition follows.

1.1.9. If x is a normal element of A and $f \in C_0(\mathrm{Sp}(x)\backslash\{0\})$ we denote by $f(x)$ the element of A corresponding to f via the embedding of $C_0(\mathrm{Sp}(x)\backslash\{0\})$ into A given by 1.1.8.

If f is a continuous function on $\boldsymbol{C}$ vanishing at 0 then it can be approximated uniformly by polynomials on any given compact set. It follows that if $\{x_n\}$ is a sequence of normal elements converging to x, then $\{f(x_n)\}$ converges to $f(x)$.

1.1.10. Let x be a self-adjoint element of A. By 1.1.5 $\mathrm{Sp}(x) \subset \boldsymbol{R}$. We write x_+ for $f_1(x)$, where $f_1(t) = t \vee 0$; x_- for $f_2(x)$, where $f_2(t) = -(t \wedge 0)$ and $|x|$ for $f_3(x)$, where $f_3(t) = |t|$. Then $x = x_+ - x_-$, $|x| = x_+ + x_-$ and $x_+ x_- = 0$. We say that x_+ and x_- are the *positive* and *negative part* of x and that $|x|$ is the *absolute value* of x. If $\mathrm{Sp}(x) \subset \boldsymbol{R}_+$ we write $x^{1/2}$ for $f_4(t)$, where $f_4(t) = t^{1/2}$. It will be shown in 1.3.3 that $\mathrm{Sp}(x^*x) \subset \boldsymbol{R}_+$ for any x in A. We will then define $|x| = (x^*x)^{1/2}$ to be the absolute value of x.

1.1.11. If $x \in A_{sa}$ and $\|x\| \leqslant 1$ then $u = x + \mathrm{i}(1 - x^2)^{1/2}$ is a normal element with $u^* = x - \mathrm{i}(1 - x^2)^{1/2}$. Since $u^*u = 1$, u is unitary. But $x = \frac{1}{2}(u + u^*)$ which shows that each element in A can be written as a linear combination of (four) unitary elements.

An elementary calculation shows that if x and y are invertible in A and $x^*x = y^*y$ then the element xy^{-1} is unitary. This is used in the proof of 1.1.12.

1.1.12. PROPOSITION. *If A is a C^*-algebra with unit then the unit ball in A is the closed convex hull of the unitary elements in A.*

Proof. If $x \in A$ and $\|x\| < 1$ the spectrum of $1 - xx^*$ is strictly positive so that the element

$$f(x, \lambda) = (1 - xx^*)^{-1/2}(1 + \lambda x)$$

exists in A and is invertible for each λ in $\boldsymbol{C}$ with $|\lambda| = 1$. Using the power series expansion $(1 - xx^*)^{-1} = \Sigma(xx^*)^n$ we see that $x^*(1 - xx^*)^{-1} = (1 - x^*x)^{-1}x^*$, whence

$$\begin{aligned} f(x, \lambda)^* f(x, \lambda) + 1 &= (1 + \bar{\lambda}x^*)(1 - xx^*)^{-1}(1 + \lambda x) + 1 \\ &= (1 - xx^*)^{-1} + (1 - x^*x)^{-1}\bar{\lambda}x^* + (1 - xx^*)^{-1}\lambda x \\ &\quad + (1 - x^*x)^{-1}. \end{aligned}$$

This expression is unchanged when exchanging x by x^* and λ by $\bar{\lambda}$ and we conclude that

$$f(x, \lambda)^* f(x, \lambda) = f(x^*, \bar{\lambda})^* f(x^*, \bar{\lambda}).$$

It follows that for each λ in $\boldsymbol{C}$ with $|\lambda| = 1$ the element $u_\lambda = f(x, \lambda) f(x^*, \bar{\lambda})^{-1}$ is unitary (cf. 1.1.11).

The function

$$u(\lambda) = (1 - xx^*)^{-1/2}(\lambda + x)(1 + \lambda x^*)^{-1}(1 - x^*x)^{1/2}$$

is holomorphic in a neighbourhood of the closed unit disc and $u(\lambda) = \lambda u_{\bar{\lambda}}$ when $|\lambda| = 1$. Moreover,

$$u(0) = (1 - xx^*)^{-1/2}x(1 - x^*x)^{1/2} = (1 - xx^*)^{-1/2}(1 - xx^*)^{1/2}x = x.$$

It follows from Cauchy's integral formula (A4, Appendix) that

$$x = (2\pi)^{-1}\int_0^{2\pi} u(e^{it})\,dt.$$

Since the measure $(2\pi)^{-1}\,dt$ on $[0, 2\pi]$ can be approximated by convex combinations of point measures, and since the elements $u(e^{it})$ are unitary in A, the open unit ball of A is contained in the closed convex hull of the unitary elements in A, from which the proposition follows.

1.1.13. If A is a C^*-algebra with unit then 1 is an extreme point in the unit ball of A. For if $1 = \frac{1}{2}(x + y)$ with x and y in A_{sa} then x commutes with y and by spectral theory $x = y = 1$. In the general case we have $1 = \frac{1}{2}[\frac{1}{2}(x + x^*) + \frac{1}{2}(y + y^*)]$ whence $\frac{1}{2}(x + x^*) = \frac{1}{2}(y + y^*) = 1$. Thus x and y are normal elements and again $x = y = 1$ from spectral theory.

Since multiplication by a unitary element is a linear isometry of A it follows from the above that every unitary in A is an extreme point in the unit ball of A. From 1.1.12 we see that the unit ball of a C^*-algebra with unit is the closed convex hull of its extreme points. This is remarkable since the unit ball is not in general compact either in the norm topology or in any other vector space topology on A.

1.1.14. Notes and remarks. The axioms for an abstract C^*-algebra were formulated in 1943 by Gelfand and Naimark [92]. With the aid of an extra axiom (namely that the spectrum of x^*x is positive for every x) they showed that any C^*-algebra is isomorphic to an algebra of operators on a Hilbert space. The theory of operator algebras had been developed during the thirties by Murray and von Neumann in a series of papers [175, 169, 170, 176, 171], dealing mainly with the weakly closed algebras (= von Neumann algebras). The name C^*-algebra was coined by Segal in [235] where the foundations for representation theory were laid. Presumably the C is meant to indicate that a C^*-algebra is a non-commutative analogue of $C(T)$, whereas the * recalls the importance of the involution.

The result in 1.1.11 is an early discovery; that in 1.1.12 is more recent [189].

1.2. Examples

1.2.1. As mentioned in 1.1.5 there is a bijective correspondence between commutative C^*-algebras and locally compact Hausdorff spaces. Non-commutative examples of C^*-algebras arise by considering the set $B(H)$ of bounded linear operators on a (complex) Hilbert space H. With the operator sum, product and norm and with the adjoint operation as involution, $B(H)$ becomes a C^*-algebra which is non-commutative when $\dim(H) > 1$. We shall study $B(H)$ and its subalgebras in some detail in the next chapter. When $\dim(H) = n < \infty$ we may identify $B(H)$ with the algebra M_n of (complex) $n \times n$ matrices.

1.2.2. Given two C^*-algebras A and B there are in general several ways of completing the algebraic tensor product $A \otimes B$ (which is an algebra with involution in a natural way) to obtain a C^*-algebra. We shall content ourselves here with the case where one of the factors is commutative so that this unpleasantness does not occur.

Let T be a locally compact Hausdorff space and A a C^*-algebra. By $C^b(T, A)$ we understand the set of bounded continuous functions x from T to A and by $C_0(T, A)$ the subset of functions x vanishing at infinity, i.e. the function $t \to \|x(t)\|$ belongs to $C_0(T)$. With pointwise sum, product and involution, and with $\|x\| = \sup \|x(t)\|$ for each x in $C_0(T, A)$ we obtain a C^*-algebra such that $C_0(T) \otimes A$ form a dense subset.

1.2.3. The simplest example of a non-commutative, infinite-dimensional C^*-algebra is probably $C_0(\boldsymbol{N}, \boldsymbol{M}_2)$. If one prefers an algebra with unit then $C(\boldsymbol{N} \cup \{\infty\}, \boldsymbol{M}_2)$ is a good example. This last algebra also has some C^*-subalgebras which are useful when trying to find counter-examples. For instance the set of sequences in $C(\boldsymbol{N} \cup \{\infty\}, \boldsymbol{M}_2)$ which tend to a diagonal matrix; the set of sequences which tend to a multiple of the identity matrix (nothing but $C_0(\boldsymbol{N}, \boldsymbol{M}_2)^{\sim}$ as defined in 1.1.3); or the set of sequences x such that $(x(\infty))_{ij} = 0$ unless $i = j = 1$.

1.2.4. Let $\{A_i \mid i \in I\}$ be a family of C^*-algebras. The set of functions x from I into $\cup A_i$ such that $x_i \in A_i$ for each i in I and such that the function $i \to \|x_i\|$ is bounded, is a C^*-algebra with pointwise sum, product and involution. We shall denote this C^*-algebra by ΠA_i and call it the *direct product* of the A_i's. Considering instead the elements in ΠA_i such that $\|x_i\| \to 0$ as $i \to \infty$ (with I as a discrete space) we obtain the *direct sum* of the A_i's, which we denote by $\oplus A_i$. When I is a finite set we may of course write $A_1 \oplus A_2 \ldots \oplus A_n$ instead of $\oplus A_i \ (= \Pi A_i)$.

If $A_i = A$ for all i in I then

$$\prod A_i = C^b(I, A) \qquad \text{and} \qquad \bigoplus A_i = C_0(I, A).$$

1.2.5. The most important non-commutative infinite-dimensional C^*-algebra is the C^*-subalgebra $\boldsymbol{C}(H)$ of $\boldsymbol{B}(H)$ consisting of the *compact operators* on H. Since $\boldsymbol{C}(H)$ is a minimal closed ideal of $\boldsymbol{B}(H)$, being the closure of the finite-dimensional operators, $\boldsymbol{C}(H)$ is *simple*, i.e. contains no non-zero closed ideals. This means that $\boldsymbol{C}(H)$ cannot be decomposed into smaller algebras, which explains its role as building block in more complicated C^*-algebras. Note that $1 \notin \boldsymbol{C}(H)$ if $\dim H = \infty$ and that $\boldsymbol{C}(H)$ is separable if H is separable.

1.2.6. Notes and remarks. The general theory of tensor products of C^*-algebras can be found in Sakai's book [231]. We return in Chapter 6 to tensor products of matrix algebras as a means to generate new algebras by an inductive limit procedure (infinite tensor products). See also 8.15.15.

1.3. Positive elements and order

1.3.1. LEMMA. *The following four conditions on an element x in A are equivalent:*

(i) *x is normal and* $\mathrm{Sp}(x) \subset \boldsymbol{R}_+$;

(ii) $x = y^2$ *with y in* A_{sa};

(iii) $x = x^*$ *and* $\|t1 - x\| \leqslant t$ *for any* $t \geqslant \|x\|$;

(iv) $x = x^*$ *and* $\|t1 - x\| \leqslant t$ *for some* $t \geqslant \|x\|$.

Proof. (i) ⇒ (ii). Using 1.1.8 we set $y = x^{1/2}$ and have $y^2 = x$. (ii) ⇒ (i). Embedding x and y in a commutative C^*-subalgebra we see that $x = x^*$ and $\mathrm{Sp}(x) \subset \boldsymbol{R}_+$. (i) ⇒ (iii). From 1.1.6 we have $\|z\| = \sup\{|\lambda| \mid \lambda \in \mathrm{Sp}(z)\}$ for each normal element z of $\tilde{A}$. Applying this to $t1 - x$ with $t \geqslant \|x\|$ we have

$$\|t1 - x\| = \sup\{|t - \lambda| \mid \lambda \in \mathrm{Sp}(x)\} \leqslant t.$$

(iii) ⇒ (iv) is immediate. (iv) ⇒ (i). If $\lambda \in \mathrm{Sp}(x)$ then $t - \lambda \in \mathrm{Sp}(t1 - x)$, whence

$$|t - \lambda| \leqslant \|t1 - x\| \leqslant t.$$

Therefore $\lambda > 0$ since $\lambda \leqslant t$.

1.3.2. The elements x of a C^*-algebra A satisfying the conditions in 1.3.1 are called *positive* (in symbols $x \geqslant 0$), and the positive part of a subset B of A is denoted by B_+.

1.3.3. THEOREM. *The set A_+ is a closed real cone in A_{sa}, and $x \in A_+$ if and only if $x = y^*y$ for some y in A.*

Proof. From 1.3.1 (iii) it is clear that A_+ is a closed subset of A_{sa} stable under multiplication with positive scalars. To prove that A_+ is a cone take x and y in A_+. By 1.3.1 (iii) we have

$$\|(\|x\| + \|y\|)1 - (x + y)\| = \|(\|x\|1 - x) + (\|y\|1 - y)\|$$
$$\leqslant \|\|x\|1 - x\| + \|\|y\|1 - y\| \leqslant \|x\| + \|y\|,$$

whence $x + y \in A_+$ by 1.3.1 (iv) since $\|x\| + \|y\| \geqslant \|x + y\|$.

Assume now that $x = y^*y$. Then $x = x^*$ so that $x = x_+ - x_-$ by 1.1.8. Moreover

$$(yx_-^{1/2})^*(yx_-^{1/2}) = x_-^{1/2}y^*yx_-^{1/2} = x_-^{1/2}(x_+ - x_-)x_-^{1/2} = -x_-^2 \in -A_+.$$

Put $yx_-^{1/2} = a + ib$ with a and b in A_{sa}. Then

$$(yx_-^{1/2})(yx_-^{1/2})^* = 2(a^2 + b^2) - (yx_-^{1/2})^*(yx_-^{1/2}) \in A_+$$

since A_+ was a cone. But, zero apart the spectrum of a product does not depend on the order of the factors (A1, Appendix), whence

$$\mathrm{Sp}(x_-^2) \subset \boldsymbol{R}_+ \cap -\boldsymbol{R}_+ = 0,$$

so that $x_- = 0$ and $x \geqslant 0$.

1.3.4. Since $A_+ - A_+ = A_{sa}$ and $A_+ \cap (-A_+) = 0$, A_{sa} becomes a partially ordered real vector space by defining $x \leqslant y$ whenever $y - x \in A_+$. When A is non-commutative A_{sa} is not a vector lattice.

1.3.5. PROPOSITION. *If $0 \leqslant x \leqslant y$ then $a^*xa \leqslant a^*ya$ for each a in A and $\|x\| \leqslant \|y\|$.*

Proof. Since $y - x = b^*b$ from 1.3.3 we have $a^*ya - a^*xa = (ba)^*(ba) \in A_+$.

Adjoining a unit to A we have $y \leqslant \|y\| 1$ from spectral theory. Then $x \leqslant \|y\| 1$, whence $\|x\| \leqslant \|y\|$.

1.3.6. PROPOSITION. *If* $1 \in A$ *and* x *and* y *are invertible elements in* A_+ *with* $x \leqslant y$ *then* $y^{-1} \leqslant x^{-1}$.

Proof. From 1.3.5 we have $y^{-1/2}xy^{-1/2} \leqslant 1$, whence $\|x^{1/2}y^{-1/2}\| \leqslant 1$ and thus $\|x^{1/2}y^{-1}x^{1/2}\| \leqslant 1$ which implies that $x^{1/2}y^{-1}x^{1/2} \leqslant 1$. By 1.3.5 $y^{-1} \leqslant x^{-1/2}1x^{-1/2} = x^{-1}$.

1.3.7. We say that a continuous real function f on an interval in $\boldsymbol{R}$ is *operator monotone* (increasing) if $x \leqslant y$ implies $f(x) \leqslant f(y)$ whenever the spectra of x and y belong to the interval of definition for f.

For each $\alpha > 0$ define f_α on $]-1/\alpha, \infty[$ by

$$f_\alpha(t) = (1 + \alpha t)^{-1}t = [1 - (1 + \alpha t)^{-1}]/\alpha.$$

Since the process of taking inverses is operator monotone decreasing by 1.3.6, it is easy to see that f_α is operator monotone increasing on $]-1/\alpha, \infty[$. The family of functions $\{f_\alpha\}$ will be used repeatedly in the sequel. Note that $f_\alpha(t) < \mathrm{Min}\{t, 1/\alpha\}$ and that $\mathrm{Lim}\, f_\alpha(t) = t$ uniformly on compact subsets of $\boldsymbol{R}$ when $\alpha \to 0$. Moreover, $f_\alpha \geqslant f_\beta$ when $\alpha \leqslant \beta$ and $f_\alpha \circ f_\beta = f_{\alpha+\beta}$ on $]-(\alpha+\beta)^{-1}, \infty[$. Finally, if $t > 0$ then $\mathrm{Lim}\, \alpha f_\alpha(t) = 1$ uniformly on compact subsets of $]0, \infty[$ when $\alpha \to \infty$.

1.3.8. PROPOSITION. *If* $0 < \beta \leqslant 1$ *the function* $t \to t^\beta$ *is operator monotone on* $\boldsymbol{R}_+$.

Proof. If $0 \leqslant x \leqslant y$ then $f_\alpha(x) \leqslant f_\alpha(y)$ with f_α as in 1.3.7. Now

$$\int_0^\infty f_\alpha(t)\alpha^{-\beta}\,d\alpha = \int_0^\infty (1 + \alpha t)^{-1}t\alpha^{-\beta}\,d\alpha$$

$$= \int_0^\infty (1 + \alpha)^{-1}t\alpha^{-\beta}t^\beta t^{-1}\,d\alpha = t^\beta \int_0^\infty (1 + \alpha)^{-1}\alpha^{-\beta}\,d\alpha = \gamma t^\beta$$

with γ in $\boldsymbol{R}_+$. For all t in $[0, \|y\|]$ and $\varepsilon > 0$ there is therefore a large n and an equidistant division $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = n$ of the interval $[0, n]$ such that

$$\left|t^\beta - (\gamma m)^{-1}n \sum_{k=1}^{m} f_{\alpha_k}(t)\alpha_k^{-\beta}\right| < \varepsilon.$$

It follows that $y^\beta - x^\beta \geqslant -2\varepsilon$, and since ε is arbitrary $x^\beta \leqslant y^\beta$.

1.3.9. PROPOSITION. *If* $0 \leqslant x \leqslant y$ *implies* $x^\beta \leqslant y^\beta$ *for some* $\beta > 1$ *and all* x, y *in a* C^**-algebra* A, *then* A *is commutative.*

Proof. By iteration we see that if the exponent β preserves order then so does β^n for every n in $\boldsymbol{N}$. Using 1.3.8 we see that the exponents which preserve order form a segment of $\boldsymbol{R}_+$. It suffices therefore to prove the proposition with $\beta = 2$.

Take x, y in A_+ and $\varepsilon > 0$. Then $x \leqslant x + \varepsilon y$ whence

$$x^2 \leqslant (x + \varepsilon y)^2 = x^2 + \varepsilon(xy + yx) + \varepsilon^2 y^2.$$

This gives $0 \leqslant xy + yx + \varepsilon y^2$ for any $\varepsilon > 0$, thus

$$(*) \qquad xy + yx \geqslant 0.$$

Set $xy = a + ib$ with a and b in A_{sa}. Clearly $a \geqslant 0$. But $(*)$ is valid for any product of positive elements and

$$(**) \qquad x(yxy) = a^2 - b^2 + i(ab + ba)$$

from which we conclude that $a^2 - b^2 \geqslant 0$.

The set of numbers $\alpha \geqslant 1$ such that $\alpha b^2 \leqslant a^2$ for all x and y in A_+ with $xy = a + ib$ is therefore non-empty. The set E is also closed, so if it was bounded it would have a largest element, say λ. Thus if x, y belongs to A_+ and $xy = a + ib$ then $a^2 - \lambda b^2 \geqslant 0$ and therefore by $(*)$

$$(***) \qquad 0 \leqslant b^2(a^2 - \lambda b^2) + (a^2 - \lambda b^2)b^2 = b^2a^2 + a^2b^2 - 2\lambda b^4.$$

From $(**)$ we now have

$$\lambda(ab + ba)^2 \leqslant (a^2 - b^2)^2,$$

that is

$$\lambda[ab^2a + ba^2b + a(bab) + (bab)a] \leqslant a^4 + b^4 - a^2b^2 - b^2a^2.$$

On the left-hand side we have $a(bab) + (bab)a \geqslant 0$ by $(*)$ and $ba^2b \geqslant \lambda b^4$ by assumption and finally, $ab^2a \geqslant 0$. Using this, and inserting $(***)$ on the right-hand side we get

$$\lambda^2 b^4 \leqslant a^4 + (1 - 2\lambda)b^4,$$

that is

$$(\lambda^2 + 2\lambda - 1)b^4 \leqslant a^4.$$

By 1.3.8 this implies that $(\lambda^2 + 2\lambda - 1)^{1/2} b^2 \leqslant a^2$ for all a and b with $a + ib = xy$ and x, y in A_+. But then $(\lambda^2 + 2\lambda - 1)^{1/2} \in E$ in contradiction with our choice of λ as the largest element. It follows that E is unbounded, whence $\alpha b^2 \leqslant a^2$ for all $\alpha \geqslant 1$, i.e. $b = 0$, and A is commutative.

1.3.10. We say that a continuous real function f on an interval in $\boldsymbol{R}$ is *operator convex* if for any two operators x, y with spectrum in this interval and any λ in $[0, 1]$

$$f(\lambda x + (1 - \lambda)y) \leqslant \lambda f(x) + (1 - \lambda)f(y).$$

We say that f is *operator concave* if $-f$ is operator convex.

1.3.11. PROPOSITION. *The functions f_α, $\alpha \geqslant 0$, the functions $t \to t^\beta$, $0 < \beta \leqslant 1$, and the functions $t \to \log(\varepsilon + t)$, $\varepsilon > 0$, are all operator concave on $\boldsymbol{R}_+$.*

Proof. If a is a positive invertible operator then from spectral theory we have for each λ in $[0, 1]$

$$[\lambda + (1 - \lambda)a]^{-1} \leqslant \lambda + (1 - \lambda)a^{-1}.$$

If x and y are positive invertible operators put $a = x^{-1/2}yx^{-1/2}$. Multiplying the inequality above with $x^{-1/2}$ from both sides, we get by 1.3.5

$$[\lambda x + (1 - \lambda)y]^{-1} \leqslant \lambda x^{-1} + (1 - \lambda)y^{-1},$$

which shows that the function $t \to t^{-1}$ is operator convex on $]0, \infty[$. It follows immediately from the formula in 1.3.7 that the functions f_α, $\alpha \geqslant 0$, are operator concave on $\boldsymbol{R}_+$.

Since operator concavity like operator monotonicity is preserved under limits (uniformly on compact subsets) and under convex combinations we see exactly as in the proof of 1.3.8 that the functions $t \to t^\beta$, $0 < \beta \leqslant 1$, are operator concave. Finally, for each $\varepsilon > 0$ put

$$g_\alpha(t) = (\alpha + 1)^{-1}(\alpha + t + \varepsilon)^{-1}(t + \varepsilon - 1) = (\alpha + 1)^{-1} - (\alpha + t + \varepsilon)^{-1}.$$

The last expression shows that g_α is operator concave for $\alpha \geqslant 0$. The first expression shows that for each $t \geqslant 0$ the function $\alpha \to g_\alpha(t)$ is integrable. An elementary calculation yields

$$\int_0^\infty g_\alpha(t)\, d\alpha = \log(\varepsilon + t)$$

and consequently the functions $t \to \log(\varepsilon + t)$, $\varepsilon > 0$ are operator concave. Incidentally, the argument also shows that the functions are operator monotone.

1.3.12. Notes and remarks. The result in 1.3.3, due to Kelley and Vaught [148], shows that the extra axiom $(x^*x \geqslant 0)$ in the original definition of a C^*-algebra (see 1.1.14) was redundant, as Gelfand and Naimark also suspected.

Operator monotone functions were characterized by Löwner [163] as being those continuous functions $f: I \to \boldsymbol{R}$ which admit a holomorphic extension $\tilde{f}$ o the upper half plane $\boldsymbol{C}_+ = \{\operatorname{Im} z > 0\}$ such that $\tilde{f}(\boldsymbol{C}_+) \subset \boldsymbol{C}_+$. It follows by a slight variation of Herglotz's formula that each function f which is operator monotone on $\boldsymbol{R}_+$ has a unique representation $f(t) = \int_0^\infty f_\alpha(t)\, d\mu(\alpha)$ for some positive measure μ on $\boldsymbol{R}_+$. The result in 1.3.9 is found in [179]. Operator convexity and monotonicity is treated in [16].