# C\*-ALGEBRAS AND THEIR AUTOMORPHISM GROUPS

GERT K. PEDERSEN

# C\*-ALGEBRAS AND THEIR AUTOMORPHISM GROUPS

GERT K. PEDERSEN

Mathematics Institute University of Copenhagen Copenhagen, Denmark

# ACADEMIC PRESS LONDON - NEW YORK SAN FRANCISCO

A Subsidiary of Harcourt Brace Jovanovich, Publishers

### ACADEMIC PRESS INC. (LONDON) LTD. 24/28 Oval Road London NW1

United States Edition published by ACADEMIC PRESS INC. 111 Fifth Avenue New York, New York 10003

Copyright © 1979 by ACADEMIC PRESS INC. (LONDON) LTD.

#### All Rights Reserved

No part of this book may be reproduced in any form by photostat, microfilm, or any other means, without written permission from the publishers

#### British Library Cataloguing in Publication Data

Pedersen, Gert K

C\*-algebras and their automorphism groups.—(London Mathematical Society. Monographs; 14 ISSN 0076-0560).

1. C\*-algebras

I. Title II. Series

512'.55 QA326

ISBN 0-12-549450-5

LCCCN 78-18028

PRINTED IN GREAT BRITAIN BY
PAGE BROS (NORWICH) LTD, MILE CROSS'LANE, NORWICH

# **Preface**

The theory of C\*-algebras is the study of operators on Hilbert space with algebraic methods. The motivating example is the spectral theorem for a normal operator (which, in effect, is nothing but Gelfand transformation applied to the algebra generated by the operator). The applications of the theory range from group representations to model quantum field theory and quantum statistical mechanics.

Already the C\*-algebra theory has grown to a size where any comprehensive treatment would result in a series of volumes more suited as a source of references than as a textbook. The material presented here has been limited by the author's knowledge and prejudice to form a somewhat manageable version. Thus the aspects of the theory concerning partially ordered vector spaces are treated in great detail. Also, since G\*-algebra theory has benefited tremendously from impulses from mathematical physics, it seemed proper to give an account which would please the C\*-physicists. Therefore the problems connected with groups of automorphisms have received special attention in this treatise. In the converse direction, the theory of von Neumann algebras, often so dominantly exposed, has here been reduced to its proper place as ancilla C\*-algebrae.

At the end of each section a few remarks are inserted, with references to the bibliography. The intention is to give the reader a rough idea of the development of the subject. Such personal comments are bound to contain errors, and the author humbly asks forgiveness from the mathematicians who have undeservedly not been mentioned.

Many people were important for the completion of this book: Richard Kadison whose work has been a constant source of inspiration for me; Daniel Kastler who provided shelter and a two months raincurtain when the work was begun in 1974; colleagues who shouldered my teaching load while I was writing; and students at the University of Copenhagen who were exposed to the first wildly incorrect drafts. It is a pleasure to record my thanks to all of them.

Copenhagen August 1978 Gert Kjærgård Pedersen

# **Contents**

| Chapter 2: Concrete C*-Algebras  2.1 Topologies on B(H)  2.2 Von Neumann's Bicommutant Theorem  2.3 Kaplansky's Density Theorem  2.4 The Up-Down Theorem  2.5 Normal Morphisms and Ideals  2.6 The Central Cover  2.7 A Generalization of Lusin's Theorem  2.8 Maximal Commutative Subalgebras  Chapter 3: Functionals and Representations  3.1 Positive Functionals  3.2 The Jordan Decomposition  3.3 The Gelfand-Naimark-Segal Construction  3.4 Commutative von Neumann Algebras  3.5 The \(\sigma\)-weak Topology on \(B(H)\)  3.6 Normal Functionals  3.7 The Universal Representation  3.8 The Enveloping von Neumann Algebra  3.9 Abstract von Neumann Algebras  3.10 Kadison's Function Representation  3.11 Semi-continuity  3.12 Multipliers  3.13 Pure States and Irreducible Representations  | v   | 3317.0    | We of      | 100      | * * *        | Se je          | face                                | Prefa |
|--|---|-----------|------------|----------|--------------|----------------|-------------------------------------|-------|
| 1.2 Examples 1.3 Positive Elements and Order 1.4 Approximate Units and Factorization Theorems 1.5 Hereditary Algebras, Ideals and Quotients  1.6 Chapter 2: Concrete C*-Algebras  2.1 Topologies on B(H) 2.2 Von Neumann's Bicommutant Theorem 2.3 Kaplansky's Density Theorem 2.4 The Up-Down Theorem 2.5 Normal Morphisms and Ideals 2.6 The Central Cover 2.7 A Generalization of Lusin's Theorem 2.8 Maximal Commutative Subalgebras  Chapter 3: Functionals and Representations 3.1 Positive Functionals 3.2 The Jordan Decomposition 3.3 The Gelfand-Naimark-Segal Construction 3.4 Commutative von Neumann Algebras 3.5 The \(\sigma\)-weak Topology on \(B(H)\) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations   |   | 13        |            | 44.      |              |                | pter 1: Abstract C*-Algebras .      | Chap  |
| 1.3 Positive Elements and Order 1.4 Approximate Units and Factorization Theorems 1.5 Hereditary Algebras, Ideals and Quotients  Chapter 2: Concrete C*-Algebras  2.1 Topologies on B(H) 2.2 Von Neumann's Bicommutant Theorem 2.3 Kaplansky's Density Theorem 2.4 The Up-Down Theorem 2.5 Normal Morphisms and Ideals 2.6 The Central Cover 2.7 A Generalization of Lusin's Theorem 2.8 Maximal Commutative Subalgebras  Chapter 3: Functionals and Representations 3.1 Positive Functionals 3.2 The Jordan Decomposition 3.3 The Gelfand-Naimark-Segal Construction 3.4 Commutative von Neumann Algebras 3.5 The σ-weak Topology on B(H) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations   | 1   |           | 5          | 4 .      | 4.3          | 1.0            | Spectral Theory                     | 1.1   |
| 1.4 Approximate Units and Factorization Theorems 1.5 Hereditary Algebras, Ideals and Quotients  Chapter 2: Concrete C*-Algebras  2.1 Topologies on B(H) 2.2 Von Neumann's Bicommutant Theorem 2.3 Kaplansky's Density Theorem 2.4 The Up-Down Theorem 2.5 Normal Morphisms and Ideals 2.6 The Central Cover 2.7 A Generalization of Lusin's Theorem 2.8 Maximal Commutative Subalgebras  Chapter 3: Functionals and Representations 3.1 Positive Functionals 3.2 The Jordan Decomposition 3.3 The Gelfand-Naimark-Segal Construction 3.4 Commutative von Neumann Algebras 3.5 The \(\sigma\)-weak Topology on \(B(H)\) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebras 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations   | 5   | T eller   |            |          | 7.           |                |                                     | 1.2   |
| Chapter 2: Concrete C*-Algebras  2.1 Topologies on B(H)  2.2 Von Neumann's Bicommutant Theorem  2.3 Kaplansky's Density Theorem  2.4 The Up-Down Theorem  2.5 Normal Morphisms and Ideals  2.6 The Central Cover  2.7 A Generalization of Lusin's Theorem  2.8 Maximal Commutative Subalgebras  Chapter 3: Functionals and Representations  3.1 Positive Functionals  3.2 The Jordan Decomposition  3.3 The Gelfand-Naimark-Segal Construction  3.4 Commutative von Neumann Algebras  3.5 The \(\sigma\)-weak Topology on \(B(H)\)  3.6 Normal Functionals  3.7 The Universal Representation  3.8 The Enveloping von Neumann Algebra  3.9 Abstract von Neumann Algebras  3.10 Kadison's Function Representation  3.11 Semi-continuity  3.12 Multipliers  3.13 Pure States and Irreducible Representations  | 6   |           |            |          | 7k           |                | Positive Elements and Order .       | 1.3   |
| Chapter 2: Concrete C*-Algebras  2.1 Topologies on B(H)  2.2 Von Neumann's Bicommutant Theorem  2.3 Kaplansky's Density Theorem  2.4 The Up-Down Theorem  2.5 Normal Morphisms and Ideals  2.6 The Central Cover  2.7 A Generalization of Lusin's Theorem  2.8 Maximal Commutative Subalgebras  Chapter 3: Functionals and Representations  3.1 Positive Functionals  3.2 The Jordan Decomposition  3.3 The Gelfand-Naimark-Segal Construction  3.4 Commutative von Neumann Algebras  3.5 The \(\sigma\)-weak Topology on \(B(H)\)  3.6 Normal Functionals  3.7 The Universal Representation  3.8 The Enveloping von Neumann Algebra  3.9 Abstract von Neumann Algebras  3.10 Kadison's Function Representation  3.11 Semi-continuity  3.12 Multipliers  3.13 Pure States and Irreducible Representations  | 11  | 1000      |            |          | rems         | tion Theor     | Approximate Units and Factoriza     | 1.4   |
| Chapter 2: Concrete C*-Algebras  2.1 Topologies on B(H)  2.2 Von Neumann's Bicommutant Theorem  2.3 Kaplansky's Density Theorem  2.4 The Up-Down Theorem  2.5 Normal Morphisms and Ideals  2.6 The Central Cover  2.7 A Generalization of Lusin's Theorem  2.8 Maximal Commutative Subalgebras  Chapter 3: Functionals and Representations  3.1 Positive Functionals  3.2 The Jordan Decomposition  3.3 The Gelfand-Naimark-Segal Construction  3.4 Commutative von Neumann Algebras  3.5 The σ-weak Topology on B(H)  3.6 Normal Functionals  3.7 The Universal Representation  3.8 The Enveloping von Neumann Algebra  3.9 Abstract von Neumann Algebras  3.10 Kadison's Function Representation  3.11 Semi-continuity  3.12 Multipliers  3.13 Pure States and Irreducible Representations   |   |           |            | 95 - 10V |              |                |                                     | 1.5   |
| 2.1 Topologies on B(H) 2.2 Von Neumann's Bicommutant Theorem 2.3 Kaplansky's Density Theorem 2.4 The Up-Down Theorem 2.5 Normal Morphisms and Ideals 2.6 The Central Cover 2.7 A Generalization of Lusin's Theorem 2.8 Maximal Commutative Subalgebras  Chapter 3: Functionals and Representations  3.1 Positive Functionals 3.2 The Jordan Decomposition 3.3 The Gelfand-Naimark-Segal Construction 3.4 Commutative von Neumann Algebras 3.5 The σ-weak Topology on B(H) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations   | 11.00                                     |           |            |          |              | The Thi        |                                     |       |
| 2.2 Von Neumann's Bicommutant Theorem 2.3 Kaplansky's Density Theorem 2.4 The Up-Down Theorem 2.5 Normal Morphisms and Ideals 2.6 The Central Cover 2.7 A Generalization of Lusin's Theorem 2.8 Maximal Commutative Subalgebras  Chapter 3: Functionals and Representations  3.1 Positive Functionals 3.2 The Jordan Decomposition 3.3 The Gelfand-Naimark-Segal Construction 3.4 Commutative von Neumann Algebras 3.5 The σ-weak Topology on B(H) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations  | 19  | • * ?     |            |          |              | 4              | pter 2: Concrete C*-Algebras .      | Chap  |
| 2.3 Kaplansky's Density Theorem 2.4 The Up-Down Theorem 2.5 Normal Morphisms and Ideals 2.6 The Central Cover 2.7 A Generalization of Lusin's Theorem 2.8 Maximal Commutative Subalgebras  Chapter 3: Functionals and Representations 3.1 Positive Functionals 3.2 The Jordan Decomposition 3.3 The Gelfand-Naimark-Segal Construction 3.4 Commutative von Neumann Algebras 3.5 The σ-weak Topology on B(H) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations   | 19  |           |            | 100      | •            |                | Topologies on B(H)                  | 2.1   |
| 2.4 The Up-Down Theorem 2.5 Normal Morphisms and Ideals 2.6 The Central Cover 2.7 A Generalization of Lusin's Theorem 2.8 Maximal Commutative Subalgebras  Chapter 3: Functionals and Representations 3.1 Positive Functionals 3.2 The Jordan Decomposition 3.3 The Gelfand-Naimark-Segal Construction 3.4 Commutative von Neumann Algebras 3.5 The σ-weak Topology on B(H) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations   | . 21                                      | uPB p     | 2 501      | 1.1      |              | eorem .        | Von Neumann's Bicommutant Th        | 2.2   |
| 2.4 The Up-Down Theorem 2.5 Normal Morphisms and Ideals 2.6 The Central Cover 2.7 A Generalization of Lusin's Theorem 2.8 Maximal Commutative Subalgebras  Chapter 3: Functionals and Representations  3.1 Positive Functionals 3.2 The Jordan Decomposition 3.3 The Gelfand-Naimark-Segal Construction 3.4 Commutative von Neumann Algebras 3.5 The σ-weak Topology on B(H) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations  | 24  |           | 798 1      | 1        |              |                | Kaplansky's Density Theorem         | 2.3   |
| 2.5 Normal Morphisms and Ideals 2.6 The Central Cover 2.7 A Generalization of Lusin's Theorem 2.8 Maximal Commutative Subalgebras  Chapter 3: Functionals and Representations  3.1 Positive Functionals 3.2 The Jordan Decomposition 3.3 The Gelfand-Naimark-Segal Construction 3.4 Commutative von Neumann Algebras 3.5 The σ-weak Topology on B(H) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations  | . 25                                      | , ulbe    | 14.5       | 70       |              |                |                                     | 2.4   |
| 2.6 The Central Cover 2.7 A Generalization of Lusin's Theorem 2.8 Maximal Commutative Subalgebras  Chapter 3: Functionals and Representations  3.1 Positive Functionals 3.2 The Jordan Decomposition 3.3 The Gelfand-Naimark-Segal Construction 3.4 Commutative von Neumann Algebras 3.5 The σ-weak Topology on B(H) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations  | A. C. | 1 P.      | the sec    |          |              |                | Normal Morphisms and Ideals         | 2.5   |
| 2.7 A Generalization of Lusin's Theorem 2.8 Maximal Commutative Subalgebras  Chapter 3: Functionals and Representations  3.1 Positive Functionals 3.2 The Jordan Decomposition 3.3 The Gelfand-Naimark-Segal Construction 3.4 Commutative von Neumann Algebras 3.5 The σ-weak Topology on B(H) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations  | 29  | TIL.      | 100        | - 40°    |              | i tirviy       | The Central Cover                   | 2.6   |
| 2.8 Maximal Commutative Subalgebras  Chapter 3: Functionals and Representations  3.1 Positive Functionals 3.2 The Jordan Decomposition 3.3 The Gelfand-Naimark-Segal Construction 3.4 Commutative von Neumann Algebras 3.5 The σ-weak Topology on B(H) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations  | . 31                                      | de re     | 1.20       | 177      |              | rem .          | A Generalization of Lusin's Theo    | 2.7   |
| 3.1 Positive Functionals 3.2 The Jordan Decomposition 3.3 The Gelfand-Naimark-Segal Construction 3.4 Commutative von Neumann Algebras 3.5 The σ-weak Topology on B(H) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations   |   | adili     | 4.4        | - 33     | 187          | as .           | Maximal Commutative Subalgebi       | 2.8   |
| 3.1 Positive Functionals 3.2 The Jordan Decomposition 3.3 The Gelfand-Naimark-Segal Construction 3.4 Commutative von Neumann Algebras 3.5 The σ-weak Topology on B(H) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations   | . 41                                      | . 17.     |            | - 1      |              | ons .          | pter 3: Functionals and Representat | Chap  |
| 3.2 The Jordan Decomposition 3.3 The Gelfand-Naimark-Segal Construction 3.4 Commutative von Neumann Algebras 3.5 The σ-weak Topology on B(H) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations  |   | inetu     | interni    | i insi   | 1000         |                | Positive Functionals                | 3.1   |
| 3.3 The Gelfand-Naimark-Segal Construction 3.4 Commutative von Neumann Algebras 3.5 The σ-weak Topology on B(H) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations   |   | io û      | الأوروا    | all r    | - agric      |                |                                     |       |
| 3.4 Commutative von Neumann Algebras 3.5 The σ-weak Topology on B(H) 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations  |   |           |            |          |              | nstruction     |                                     |       |
| <ul> <li>3.5 The σ-weak Topology on B(H)</li> <li>3.6 Normal Functionals</li> <li>3.7 The Universal Representation</li> <li>3.8 The Enveloping von Neumann Algebra</li> <li>3.9 Abstract von Neumann Algebras</li> <li>3.10 Kadison's Function Representation</li> <li>3.11 Semi-continuity</li> <li>3.12 Multipliers</li> <li>3.13 Pure States and Irreducible Representations</li> </ul>   |   |           |            |          |              |                |                                     |       |
| 3.6 Normal Functionals 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations   |   |           |            |          |              |                |                                     |       |
| 3.7 The Universal Representation 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations  |   |           |            |          |              |                |                                     | 3.6   |
| 3.8 The Enveloping von Neumann Algebra 3.9 Abstract von Neumann Algebras 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations   |   |           |            |          |              |                | The Universal Representation        | 3.7   |
| 3.9 Abstract von Neumann Algebras  |   | all ta    | say I.     | ize.     | la erro      | lgebra .       |                                     | 3.8   |
| 3.10 Kadison's Function Representation 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations  | 65  | on the    | EU 20      | History  | 15-L. I      | united make    | Abstract von Neumann Algebras       | 3.9   |
| 3.11 Semi-continuity 3.12 Multipliers 3.13 Pure States and Irreducible Representations   | . 69                                      |           | Tro-       | O rot    | LAUGIUS      | on .           | Kadison's Function Representat      | 3.10  |
| 3.12 Multipliers 3.13 Pure States and Irreducible Representations  | . 73                                      | I lake to | DIT.       | 210x161  | 1100         | Propert relati | Semi-continuity                     |       |
| 3.13 Pure States and Irreducible Representations   | . 78                                      | ninsterin | 110.6      | 65.6     | dire         | Laterary)      | Multipliers                         | 3.12  |
| All and Abidian Symptons and A | . 84                                      | Grou      | to attu    | ouk t    | ıs .         | esentation     | Pure States and Irreducible Repr    | 3.13  |
| Chapter 4: Decomposition Theory  | gorish. Ley                               | abole in  | ( Mile     |          | ev2 -n s     | . small        |                                     | 293   |
|  | 91  |           |            |          |              |                | oter 4: Decomposition Theory .      | Chap  |
| 4.1 The Spectra of C*-Algebras   | . 91                                      | to val    | 47         | TEN      | during their | Edg Brood      | The Spectra of C*-Algebras.         | 4.1   |
|  |   | ing le    | Date of Co | 76-965   | AND E        | Soft Broken    |                                     | 4.2   |
|  |   |           |            |          |              |                |                                     | 4.3   |

viii CONTENTS

| 4.4         | The Dauns-Hofmann Theorem  |          |            |         |         |                   |                        |         | 106  |
|-------------|--|----------|------------|---------|---------|-------------------|------------------------|---------|------|
| 4.5         | Borel *-Algebras   |          | . "        |         |         | *                 |                        |         | 109  |
| 4.6         | Standard Algebras  |          |            |         |         |                   |                        |         | 117  |
| 4.7         | Borel Structures on the Spectrum   |          |            |         |         |                   |                        |         | 123  |
| 4.8         | Disintegration on the Factor Spect   |          | e lei      |         |         |                   |                        | 5       | 126  |
| 4.9         | Disintegration on the State Space  |          |            | •       |         |                   |                        |         | 130  |
| 4.10        | Standard Measures  |          |            |         | •       |                   | <i>;</i>               | levil I | 135  |
| 4.11        | Direct Integrals of von Neumann A  | Alaab    |            | •       |         | •                 | * %                    | •       | 140  |
| 4.11        |  |          | as         |         | •       | •                 |                        |         | 146  |
| 4.12        | Direct Integrals of Representations  | 5        |            | ٠.      | •       | \$ 1 <sub>0</sub> | 9                      | *       | 140  |
| Chap        | pter 5: Weights and Traces .   |          |            |         | . %     |                   |                        | •       | 151  |
| 5.1         | Weights  |          |            | 1       |         |                   | 3                      |         | 151  |
| 5.2         | Traces   |          | 7          |         |         |                   |                        |         | 155  |
| 5.3         | The Radon-Nikodym Theorem for  | Trace    | S          |         | 1 -     |                   |                        |         | 159  |
| 5.4         | Semi-finite von Neumann Algebras   |          |            |         |         |                   | har trost              | 7. 4    | 166  |
| 5.5         | Type I von Neumann Algebras  |          |            |         | i .     | T S               | - :11-                 | ÀT.     | 171  |
| 5.6         | The Minimal Dense Ideal .  |          |            | CHIPS.  |         |                   | ANT THE REAL PROPERTY. | h.      | 175  |
| 5.7         | Borel Sets in the Factor Spectrum  |          |            |         |         | A TA              |                        | 1       | 179  |
|             | Disintegration of Weights and Trace  |          |            | 2-      |         |                   |                        |         | 186  |
| 5.8         | Disintegration of weights and Trace  | CS       |            |         |         |                   |                        | Inla    | 100  |
| Chaj        | pter 6: Type I C*-Algebras   |          |            | • 1     |         |                   |                        | èr.     | 191  |
| 6.1         | Abelian Elements   |          | , Carlo    | , K.    |         |                   |                        |         | 191  |
| 6.2         | The state of the s |          |            |         |         |                   |                        | 30      | .197 |
| 6.3         | Borel *-Algebras of Type I   |          |            |         |         |                   |                        |         | 201  |
| 6.4         | Glimm Algebras .   | е е      | Total      |         |         |                   |                        |         | 205  |
| 6.5         | Product States of Glimm Algebras   |          |            |         |         |                   | Bur . 3                |         | 208  |
| Agreement 1 |  | Se i più |            |         |         |                   |                        |         | 215  |
| 6.6         |  |          |            |         |         |                   | vio 1)                 |         | 218  |
| 6.7         |  | 148 148  | y is a     | * 14X   | A PARTY |                   | •                      |         | 222  |
| 6.8         | Glimm's Theorem  |          | e Light    | Ž.      |         |                   |                        |         |      |
| Char        |  | . 77111  |            |         |         |                   |                        | A       | 227  |
| 18          |  | 9 3      |            |         | ATT OF  | in test           | 77730                  | ň.      |      |
| 7.1         | Locally Compact Groups .   |          | r att      | Def 19  |         | · a are           | 10,1. 20               | •       | 227  |
| 7.2         | The Regular Representation   |          | 1420       | •       | 100     | iel M             |                        | •       | 233  |
| 7.3         | Amenable Groups  |          |            |         |         |                   |                        |         | 239  |
| 7.4         | Dynamical Systems  |          | H pa j     | · (1)   | HOUSE   | . 250             | E TO                   |         | 244  |
| 7.5         | B*-Dynamical Systems .   |          | •          | ٠, ١    |         | MILET.            |                        |         | 250  |
| 7.6         |  | •        | 0/11       | T. Liby | y 18 1  | 1000              |                        |         | 254  |
| 7.7         | Regular Representations of Crosse  | d Pro    | ducts      | 192 1   | . 7-89  | 100° ×            |                        |         | 260  |
| 7.8         | Crossed Products with Abelian Gr   | oups     | distrib    | .000    | htt: 14 | 404               |                        | 101     | 267  |
| 7.9         | Duality Theory for Crossed Produc  | cts      | Project of | 495-1   | alon    | a sect            | in the                 |         | 273  |
| 7.10        | W*-Crossed Products with Abelian   |          |            |         |         | und.              | s lead                 |         | 278  |
| 7.11        | W*-Crossed Products with Discret   | e Gro    | ups        |         |         | 215               | logista                | M.      | 284  |
| 7.12        | Large Groups of Automorphisms  |          | 100 e10    | SALD:   | to Lity | r. ell            | 12.00                  |         | 289  |
| 7.13        | Asymptotically Abelian Systems   |          | 7          |         |         |                   |                        | 201     | 294  |
| 10          |  |          | · Crr      | soul F  | authe   | PERIO             | 40.0                   | 7530    |      |
| Chap        | ter 8: Spectral Theory for Automorph   | hism (   | Group      | dogte   | 45 1    | s silv            | eg S au                | 1       | 299  |
| 8.1         | Spectral Subspaces and Arveson Sp  | pectru   | m          |         |         | ister             | B Helt                 | ag.     | 300  |
| 8.2         | Composition of Representations   |          |            | State   | Pure    | dis.              | utulan                 | O.S     | 307  |
|             |  |          |            | C.      |         |                   | 100                    |         |      |

|          |                                  |         |     | 3.5 |     |     |   |   |     |
|----------|----------------------------------|---------|-----|-----|-----|-----|---|---|-----|
|          | C                                | ONTEN   | TS  |     | 4   |     |   |   | ix  |
|          |                                  |         |     |     | 5   |     |   |   |     |
| 8.3      | The Minimal Positive Represen    | tation  |     |     |     |     |   |   | 311 |
| 8.4      | Spectrum Conditions              |         |     |     |     |     |   |   | 318 |
| 8.5      | Uniformly Continuous Represe     | ntation | as. |     |     |     |   |   | 323 |
| 8.6      | Derivations                      |         |     | 4   |     |     |   |   | 327 |
| 8.7      | Derivable Automorphisms .        |         |     |     |     |     |   |   | 336 |
| 8.8      | Borchers and Connes Spectrum     |         |     | 4   |     |     |   |   | 340 |
| 8.9      | Inner Automorphisms .            |         |     |     |     |     |   |   | 345 |
| 8 10     | Connes Spectrum and Compact      | Grou    | กร  |     |     |     | P |   | 348 |
| 11       | Connes Spectrum and Crossed      |         |     |     |     | 3.5 |   | • | 355 |
| 8.12     | The KMS Condition .              | 1000    | ·   |     | •   | •   |   | • | 363 |
| 8.13     | The Tomita-Takesaki Theory       |         |     |     | •   |     |   | • | 371 |
| 8.14     | The Modular Group                |         |     |     | •   | •   |   | • | 378 |
| 8.15     |                                  | abeaa   | 7.0 | •   | •   |     | • |   |     |
| 0.13     | A Classification of Type III Aig | coras   | (€) |     |     | •   | • |   | 387 |
| Dafar    | rences                           |         |     |     |     |     |   |   | 206 |
| Kelei    | ences                            |         | •   |     | •   |     | • | • | 395 |
| <u> </u> | ari.                             |         |     |     | - 1 |     |   |   | 400 |
| Appe     | naix .,                          | •       | . • |     | ٠   |     |   |   | 409 |
| cL:-     | ct Index                         |         |     |     |     |     |   |   | 413 |
| NIIOIP   | THE PROPERTY.                    |         |     |     |     |     |   |   | 414 |

### Chapter 1

# Abstract C\*-algebras

# 1.1 Spectral theory

1.1.1. A  $C^*$ -algebra is a complex Banach algebra A with an involution, \*, satisfying  $||x^*x|| = ||x||^2$  for all x in A. Since  $||x^*x|| \le ||x^*|| ||x||$  we have  $||x|| \le ||x^*||$  for each x in A, whence  $||x|| = ||x^*||$ , so that the involution is isometric. An element x in A is normal if it commutes with its adjoint  $x^*$ ; and it is self-adjoint if  $x = x^*$ . The self-adjoint part of a subset B of A is denoted by  $B_{sa}$ . For each x in A the elements  $\frac{1}{2}(x + x^*)$  and  $-\frac{1}{2}i(x - x^*)$  (the real resp. imaginary part of x) belong to  $A_{sa}$ . It follows that  $A_{sa}$  is a closed real subspace of A and that each element x in A has a unique decomposition x = y + iz with y and z in  $A_{sa}$ .

1.1.2. In general a  $C^*$ -algebra need not have a unit. If however, the  $C^*$ -algebra A has a unit (denoted by  $1_A$ , or just 1 when no confusion may arise) and  $A \neq 0$ , then  $1_A^* = 1_A$  and  $||1_A|| = 1$ .

If  $1 \in A$  we say that an element u in A is unitary if  $u^*u = uu^* = 1$ . Note that each unitary is normal and has norm 1.

1.1.3. PROPOSITION. For each  $C^*$ -algebra A there is a  $C^*$ -algebra  $\tilde{A}$  with unit containing A as a closed ideal. If A has no unit then  $\tilde{A}/A = C$ .

**Proof.** Let  $\pi$  denote the left regular representation of A as operators on itself, i.e.  $\pi(x)y = xy$  for all x and y in A. It is clear that  $\pi$  is a homomorphism and that  $\|\pi(x)\| \le \|x\|$ . But since

$$||x||^2 = ||xx^*|| = ||\pi(x)x^*|| \le ||\pi(x)|| \, ||x^*||$$

we see that  $\pi$  is an isometry. Let 1 denote the identity operator on A and let  $\tilde{A}$  be the algebra of operators on A of the form  $\pi(x) + \alpha 1$  with x in A and  $\alpha$  in C. Since  $\pi(A)$  is complete and  $\tilde{A}/\pi(A) = C$ ,  $\tilde{A}$  is also complete. With the involution defined by  $(\pi(x) + \alpha 1)^* = \pi(x^*) + \bar{\alpha}1$ ,  $\tilde{A}$  becomes a  $C^*$ -algebra

since for each  $\varepsilon > 0$  there is a y in A with ||y|| = 1 such that

$$\|\pi(x) + \alpha 1\|^2 \le \varepsilon + \|(x + \alpha)y\|^2$$

$$= \varepsilon + \|y^*(x^* + \bar{\alpha})(x + \alpha)y\|_1 \le \varepsilon + \|(x^* + \bar{\alpha})(x + \alpha)y\|_2$$

$$\le \varepsilon + \|(\pi(x) + \alpha 1)^*(\pi(x) + \alpha 1)\|_2$$

1.1.4. For each x in a  $C^*$ -algebra A we define the spectrum of x in A (written  $\operatorname{Sp}_A(x)$ ) as the set of complex numbers  $\lambda$  such that  $\lambda 1 - x$  is not invertible in  $\tilde{A}$ . Note that  $0 \in \operatorname{Sp}_A(x)$  whenever  $A \neq \tilde{A}$ . By straightforward computations it follows that if  $\lambda \neq 0$ , then  $\lambda \notin \operatorname{Sp}_A(x)$  if and only if there is a y in A such that  $xy \neq yx = \lambda^{-1}x + \lambda y$  (corresponding to the fact that  $\lambda^{-1} - y = (\lambda 1 - x)^{-1}$  in  $\tilde{A}$ ).

If  $x \in A_{sa}$  and v(x) is the spectral radius of x then by repeated use of the equality  $||x^2|| = ||x||^2$  we obtain

$$v(x) = \text{Lim} \|x^{2n}\|^{2^{-n}} = \|x\|.$$

If x is just normal, then from the preceding,

$$v(x)^{2} \leq ||x||^{2} = ||x^{*}x|| = \operatorname{Lim} ||(x^{*}x)^{n}||^{n-1}$$
  
$$\leq \operatorname{Lim}(||(x^{*})^{n}|| ||x^{n}||)^{n-1} = v(x)^{2},$$

whence again v(x) = ||x||.

1.1.5. LEMMA. If  $x \in A_{su}$  then  $Sp(x) \subset R$ . If  $1 \in A$  and u is unitary, Sp(u) is contained in the unit circle.

**Proof.** If  $\lambda \in \operatorname{Sp}(u)$  then  $\lambda^{-1} \in \operatorname{Sp}(u^{-1})$ . Since  $u^{-1} = u^*$  we have  $|\lambda| \le 1$  and  $|\lambda^{-1}| \le 1$ , whence  $|\lambda| = 1$  which proves the second assertion in the lemma. Take now x in  $A_{sa}$ . The power series  $\sum (n!)^{-1} (ix)^n$  converges in  $\tilde{A}$  to an element  $\exp(ix)$  which is unitary (in  $\tilde{A}$ ) since

$$\exp(ix)^* = \exp(-ix) = \exp(ix)^{-1}.$$

If  $\lambda \in \operatorname{Sp}(x)$  then  $\exp(i\lambda) \in \operatorname{Sp}(\exp(ix))$  by computation, whence  $|\exp(i\lambda)| = 1$  by the first part of the proof. It follows that  $\operatorname{Sp}(x) \subset R$  as desired.

1.1.6. Let A be a commutative Banach algebra. The spectrum  $\hat{A}$  of A is the set of non-zero homomorphisms of A onto C. Each element in  $\hat{A}$  belongs to the unit ball of the dual  $A^*$  of A, and since  $\hat{A} \cup \{0\}$  is the weak\* closed subset of  $A^*$  consisting of functionals t such that t(xy) = t(x)t(y) for all x, y in A, we see that  $\hat{A}$  is a locally compact Hausdorff space in the weak\* topology. The Gelfand transform on A is the homomorphism  $x \to \hat{x}$  of A into  $C_0(\hat{A})$  given by  $\hat{x}(t) = t(x)$  for all x in A and t in  $\hat{A}$ .

1.1.7. THEOREM. If A is a commutative C\*-algebra then the Gelfand transform is a \*-preserving isometry of A onto  $C_0(\hat{A})$ .

**Proof.** If  $t \in \hat{A}$  and  $x \in A$  then ker t is a maximal ideal of A, whence  $t(x) \in \operatorname{Sp}(x)$  (and conversely, if  $\lambda \in \operatorname{Sp}(x) \setminus \{0\}$  then  $\lambda = t(x)$  for some t in  $\hat{A}$ ). If therefore  $x = x^*$  then  $t(x) \in \mathbf{R}$  by 1.1.5. It follows that  $t(x^*) = t(x)$  for each x in A which shows that the map  $x \to \hat{x}$  is \*-preserving (using complex conjugation of functions as involution in  $C_0(\hat{A})$ ). Moreover,  $\|\hat{x}\|$  is the spectral radius of x, whence  $\|\hat{x}\| = \|x\|$  by 1.1.4 as each x in A is normal. Thus  $x \to \hat{x}$  is a \*-preserving isometry of A into  $C_0(\hat{A})$  and since the set of functions  $\{\hat{x} \mid x \in A\}$  separates points in  $\hat{A}$  and does not all vanish at any point we conclude from the Stone–Weierstrass theorem that the image of A is all of  $C_0(\hat{A})$ .

1.1.8. PROPOSITION. Let x be a normal element of a  $C^*$ -algebra A, and let B denote the smallest  $C^*$ -subalgebra of A containing x. Then  $B = C_0(\operatorname{Sp}_A(x)\setminus\{0\})$  and  $\operatorname{Sp}_A(x)\setminus\{0\} = \operatorname{Sp}_B(x)\setminus\{0\}$ .

*Proof.* Since B is a singly generated commutative  $C^*$ -algebra it follows that  $\hat{B} = \operatorname{Sp}_B(x) \setminus \{0\}$ , whence  $B = C_0(\operatorname{Sp}_B(x) \setminus \{0\})$  by 1.1.7 (suppressing the isomorphism). If therefore  $\lambda \in \operatorname{Sp}_B(x) \setminus \{0\}$  there is for each  $\varepsilon > 0$  an element b in B with ||b|| = 1 such that  $||\lambda b - xb|| < \varepsilon$ . This shows that  $\lambda 1 - x$  is not invertible in  $\tilde{A}$ , thus  $\lambda \in \operatorname{Sp}_A(x)$ . It is immediate from 1.1.4 that if  $\lambda \notin \operatorname{Sp}_B(x)$  and  $\lambda \neq 0$  then  $\lambda \notin \operatorname{Sp}_A(x)$ , and the proposition follows.

1.1.9. If x is a normal element of A and  $f \in C_0(\mathrm{Sp}(x) \setminus \{0\})$  we denote by f(x) the element of A corresponding to f via the embedding of  $C_0(\mathrm{Sp}(x) \setminus \{0\})$  into A given by 1.1.8.

If f is a continuous function on C vanishing at 0 then it can be approximated uniformly by polynomials on any given compact set. It follows that if  $\{x_n\}$  is a sequence of normal elements converging to x, then  $\{f(x_n)\}$  converges to f(x).

1.1.10. Let x be a self-adjoint element of A. By 1.1.5 Sp(x)  $\subset \mathbb{R}$ . We write  $x_+$  for  $f_1(x)$ , where  $f_1(t) = t \vee 0$ ;  $x_-$  for  $f_2(x)$ , where  $f_2(t) = -(t \wedge 0)$  and |x| for  $f_3(x)$ , where  $f_3(t) = |t|$ . Then  $x = x_+ - x_-$ ,  $|x| = x_+ + x_-$  and  $x_+x_- = 0$ . We say that  $x_+$  and  $x_-$  are the positive and negative part of x and that |x| is the absolute value of x. If Sp(x)  $\subset \mathbb{R}_+$  we write  $x^{1/2}$  for  $f_4(t)$ , where  $f_4(t) = t^{1/2}$ . It will be shown in 1.3.3 that Sp( $x^*x$ )  $\subset \mathbb{R}_+$  for any x in A. We will then define  $|x| = (x^*x)^{1/2}$  to be the absolute value of x.

1.1.11. If  $x \in A_{sa}$  and  $||x|| \le 1$  then  $u = x + i(1 - x^2)^{1/2}$  is a normal element with  $u^* = x - i(1 - x^2)^{1/2}$ . Since  $u^*u = 1$ , u is unitary. But  $x = \frac{1}{2}(u + u^*)$  which shows that each element in A can be written as a linear combination of (four) unitary elements.

An elementary calculation shows that if x and y are invertible in A and  $x^*x = y^*y$  then the element  $xy^{-1}$  is unitary. This is used in the proof of 1.1.12.

1.1.12. PROPOSITION. If A is a  $C^*$ -algebra with unit then the unit ball in A is the closed convex hull of the unitary elements in A.

*Proof.* If  $x \in A$  and ||x|| < 1 the spectrum of  $1 - xx^*$  is strictly positive so that the element

$$f(x, \lambda) = (1 - xx^*)^{-1/2}(1 + \lambda x)$$

exists in A and is invertible for each  $\lambda$  in **C** with  $|\lambda| = 1$ . Using the power series expansion  $(1 - xx^*)^{-1} = \Sigma(xx^*)^n$  we see that  $x^*(1 - xx^*)^{-1} = (1 - x^*x)^{-1}x^*$ , whence

$$f(x,\lambda)^* f(x,\lambda) + 1 = (1 + \overline{\lambda}x^*)(1 - xx^*)^{-1}(1 + \lambda x) + 1$$
$$= (1 - xx^*)^{-1} + (1 - x^*x)^{-1}\overline{\lambda}x^* + (1 - xx^*)^{-1}\lambda x$$
$$+ (1 - x^*x)^{-1}.$$

This expression is unchanged when exchanging x by  $x^*$  and  $\lambda$  by  $\lambda$  and we conclude that

$$f(x,\lambda)*f(x,\lambda) = f(x*,\overline{\lambda})*f(x*,\overline{\lambda}).$$

It follows that for each  $\lambda$  in C with  $|\lambda| = 1$  the element  $u_{\lambda} = f(x, \lambda)f(x^*, \overline{\lambda})^{-1}$  is unitary (cf. 1.1.11).

The function

$$u(\lambda) = (1 - xx^*)^{-1/2}(\lambda + x)(1 + \lambda x^*)^{-1}(1 - x^*x)^{1/2}$$

is holomorphic in a neighbourhood of the closed unit disc and  $u(\lambda) = \lambda u_{\lambda}$  when  $|\lambda| = 1$ . Moreover,

$$u(0) = (1 - xx^*)^{-1/2}x(1 - x^*x)^{1/2} = (1 - xx^*)^{-1/2}(1 - xx^*)^{1/2}x = x.$$

It follows from Cauchy's integral formula (A4, Appendix) that

$$x = (2\pi)^{-1} \int_{0}^{2\pi} u(e^{it}) dt.$$

Since the measure  $(2\pi)^{-1} dt$  on  $[0, 2\pi]$  can be approximated by convex combinations of point measures, and since the elements  $u(e^{it})$  are unitary in A, the open unit ball of A is contained in the closed convex hull of the unitary elements in A, from which the proposition follows.

1.1.13. If A is a C\*-algebra with unit then 1 is an extreme point in the unit ball of A. For if  $1 = \frac{1}{2}(x + y)$  with x and y in  $A_{sa}$  then x commutes with y and by spectral theory x = y = 1. In the general case we have  $1 = \frac{1}{2}[\frac{1}{2}(x + x^*) + \frac{1}{2}(y + y^*)]$  whence  $\frac{1}{2}(x + x^*) = \frac{1}{2}(y + y^*) = 1$ . Thus x and y are normal elements and again x = y = 1 from spectral theory.

Since multiplication by a unitary element is a linear isometry of A it follows from the above that every unitary in A is an extreme point in the unit ball of A. From 1.1.12 we see that the unit ball of a  $C^*$ -algebra with unit is the closed convex hull of its extreme points. This is remarkable since the unit ball is not in general compact either in the norm topology or in any other vector space topology on A.

1.1.14. Notes and remarks. The axioms for an abstract  $C^*$ -algebra were formulated in 1943 by Gelfand and Naimark [92]. With the aid of an extra axiom (namely that the spectrum of  $x^*x$  is positive for every x) they showed that any  $C^*$ -algebra is isomorphic to an algebra of operators on a Hilbert space. The theory of operator algebras had been developed during the thirties by Murray and von Neumann in a series of papers [175, 169, 170, 176, 171], dealing mainly with the weakly closed algebras (= von Neumann algebras). The name  $C^*$ -algebra was coined by Segal in [235] where the foundations for representation theory were laid. Presumably the C is meant to indicate that a  $C^*$ -algebra is a non-commutative analogue of C(T), whereas the \* recalls the importance of the involution.

The result in 1.1.11 is an early discovery; that in 1.1.12 is more recent [189].

# 1.2. Examples

- 1.2.1. As mentioned in 1.1.5 there is a bijective correspondence between commutative  $C^*$ -algebras and locally compact Hausdorff spaces. Noncommutative examples of  $C^*$ -algebras arise by considering the set B(H) of bounded linear operators on a (complex) Hilbert space H. With the operator sum, product and norm and with the adjoint operation as involution, B(H) becomes a  $C^*$ -algebra which is non-commutative when  $\dim(H) > 1$ . We shall study B(H) and its subalgebras in some detail in the next chapter. When  $\dim(H) = n < \infty$  we may identify B(H) with the algebra  $M_n$  of (complex)  $n \times n$  matrices.
- 1.2.2. Given two  $C^*$ -algebras A and B there are in general several ways of completing the algebraic tensor product  $A \otimes B$  (which is an algebra with involution in a natural way) to obtain a  $C^*$ -algebra. We shall content ourselves here with the case where one of the factors is commutative so that this unpleasantness does not occur.

Let T be a locally compact Hausdorff space and A a  $C^*$ -algebra. By  $C^b(T,A)$  we understand the set of bounded continuous functions x from T to A and by  $C_0(T,A)$  the subset of functions x vanishing at infinity, i.e. the function  $t \to ||x(t)||$  belongs to  $C_0(T)$ . With pointwise sum, product and involution, and with  $||x|| = \sup ||x(t)||$  for each x in  $C_0(T,A)$  we obtain a  $C^*$ -algebra such that  $C_0(T) \otimes A$  form a dense subset.

- 1.2.3. The simplest example of a non-commutative, infinite-dimensional  $C^*$ -algebra is probably  $C_0(N, M_2)$ . If one prefers an algebra with unit then  $C(N \cup \{\infty\}, M_2)$  is a good example. This last algebra also has some  $C^*$ -subalgebras which are useful when trying to find counter-examples. For instance the set of sequences in  $C(N \cup \{\infty\}, M_2)$  which tend to a diagonal matrix; the set of sequences which tend to a multiple of the identity matrix (nothing but  $C_0(N, M_2)^{\sim}$  as defined in 1.1.3); or the set of sequences x such that  $(x(x))_{ij} = 0$  unless i = j = 1.
- 1.2.4. Let  $\{A_i \mid i \in I\}$  be a family of  $C^*$ -algebras. The set of functions x from I into  $\bigcup A_i$  such that  $x_i \in A_i$  for each i in I and such that the function  $i \to ||x_i||$  is bounded, is a  $C^*$ -algebra with pointwise sum, product and involution. We shall denote this  $C^*$ -algebra by  $\Pi$   $A_i$  and call it the direct product of the  $A_i$ 's. Considering instead the elements in  $\Pi$   $A_i$  such that  $||x_i|| \to 0$  as  $i \to \infty$  (with I as a discrete space) we obtain the direct sum of the  $A_i$ 's, which we denote by  $\bigoplus A_i$ . When I is a finite set we may of course write  $A_1 \bigoplus A_2 \ldots \bigoplus A_n$  instead of  $\bigoplus A_i (= \prod A_i)$ .

If  $A_i = A$  for all i in I then

$$\prod A_i = C^b(I, A)$$
 and  $\bigoplus A_i = C_0(I, A)$ .

- 1.2.5. The most important non-commutative infinite-dimensional  $C^*$ -algebra is the  $C^*$ -subalgebra C(H) of B(H) consisting of the compact operators on H. Since C(H) is a minimal closed ideal of B(H), being the closure of the finite-dimensional operators, C(H) is simple, i.e. contains no non-zero closed ideals. This means that C(H) cannot be decomposed into smaller algebras, which explains its role as building block in more complicated  $C^*$ -algebras. Note that  $1 \notin C(H)$  if dim  $H = \infty$  and that C(H) is separable if H is separable.
- 1.2.6. Notes and remarks. The general theory of tensor products of C\*-algebras can be found in Sakai's book [231]. We return in Chapter 6 to tensor products of matrix algebras as a means to generate new algebras by an inductive limit procedure (infinite tensor products). See also 8.15.15.

completing the algebraic tensor product in @ B (which is an algebra with modulice in a natural way) to obtain a C. sincbra. We shall content ourselves

# 1.3. Positive elements and order at add to und arodin again the area

- 1.3.1. LEMMA. The following four conditions on an element x in A are equivalent:
- (i) x is normal and  $Sp(x) \subset \mathbf{R}_+$ ;
- (ii)  $x = y^2$  with y in  $A_{sa}$ ; whereas more drive  $(X_1, X_2, X_3)$  or expected first f = x
  - (iii)  $x = x^*$  and  $||t1 x|| \le t$  for any  $t \ge ||x||$ ;
  - (iv)  $x = x^*$  and  $||t1 x|| \le t$  for some  $t \ge ||x||$ .

*Proof.* (i)  $\Rightarrow$  (ii). Using 1.1.8 we set  $y = x^{1/2}$  and have  $y^2 = x$ . (ii)  $\Rightarrow$  (i). Embedding x and y in a commutative C\*-subalgebra we see that  $x = x^*$  and  $Sp(x) \subset \mathbf{R}_+$ . (i)  $\Rightarrow$  (iii). From 1.1.6 we have  $||z|| = \sup\{|\lambda| \mid \lambda \in Sp(z)\}$  for each normal element z of  $\tilde{A}$ . Applying this to  $t1 \leftarrow x$  with  $t \geqslant ||x||$  we have

$$||t1 - x|| = \sup\{|t - \lambda| \mid \lambda \in \operatorname{Sp}(x)\} \leq t.$$

(iii)  $\Rightarrow$  (iv) is immediate. (iv)  $\Rightarrow$  (i). If  $\lambda \in \operatorname{Sp}(x)$  then  $t - \lambda \in \operatorname{Sp}(t1 - x)$ , whence  $|t - \lambda| \le |t - x| \le t$ .

Therefore  $\lambda > 0$  since  $\lambda \leq t$ .

1.3.2. The elements x of a  $C^*$ -algebra A satisfying the conditions in 1.3.1 are called *positive* (in symbols  $x \ge 0$ ), and the positive part of a subset B of A is denoted by  $B_+$ .

1.3.3. THEOREM. The set  $A_+$  is a closed real cone in  $A_{>a}$ , and  $x \in A_+$  if and only if  $x = y^*y$  for some y in A:

**Proof.** From 1.3.1 (iii) it is clear that  $A_+$  is a closed subset of  $A_{sa}$  stable under multiplication with positive scalars. To prove that  $A_+$  is a cone take x and y in  $A_+$ . By 1.3.1 (iii) we have

$$\begin{aligned} \|(\|x\| + \|y\|)1 - (x + y)\| &= \|(\|x\|1 - x) + (\|y\|1 - y)\| \\ &\leq \|\|x\|1 - x\| + \|\|y\|1 - y\| \leq \|x\| + \|y\|, \end{aligned}$$

whence  $x + y \in A_+$  by 1.3.1 (iv) since  $||x|| + ||y|| \ge ||x + y||$ .

Assume now that  $x = y^*y$ . Then  $x = x^*$  so that  $x = x_+ - x_-$  by 1.1.8. Moreover

$$(yx_{-}^{1/2})^*(yx_{-}^{1/2}) = x_{-}^{1/2}y^*yx_{-}^{1/2} = x_{-}^{1/2}(x_{+} - x_{-})x_{-}^{1/2} = -x_{-}^2 \in -A_{+}.$$

Put  $yx_{-}^{1/2} = a + ib$  with a and b in  $A_{sa}$ . Then

$$(vx_{-}^{1/2})(vx_{-}^{1/2})^* = 2(a^2 + b^2) - (vx_{-}^{1/2})^*(vx_{-}^{1/2}) \in A_+$$

since  $A_+$  was a cone. But, zero apart the spectrum of a product does not depend on the order of the factors (A1, Appendix), whence

$$\mathrm{Sp}(x_-^2) \subset \mathbf{R}_+ \cap -\mathbf{R}_+ = 0,$$

so that  $x_{-} = 0$  and  $x \ge 0$ .

1.3.4. Since  $A_+ - A_+ = A_{sa}$  and  $A_+ \cap (-A_+) = 0$ ,  $A_{sa}$  becomes a partially ordered real vector space by defining  $x \le y$  whenever  $y - x \in A_+$ . When A is non-commutative  $A_{sa}$  is not a vector lattice.

1.3.5. PROPOSITION. If  $0 \le x \le y$  then  $a*xa \le a*ya$  for each a in A and  $||x|| \le ||y||$ .

*Proof.* Since y - x = b\*b from 1.3.3 we have  $a*ya - a*xa = (ba)*(ba) \in A_+$ . Adjoining a unit to A we have  $y \le ||y|| 1$  from spectral theory. Then  $x \le ||y|| 1$ , whence  $||x|| \le ||y||$ .

1.3.6. PROPOSITION. If  $1 \in A$  and x and y are invertible elements in  $A_+$  with  $x \le y$  then  $y^{-1} \le x^{-1}$ .

*Proof.* From 1.3.5 we have  $y^{-1/2}xy^{-1/2} \leqslant 1$ , whence  $||x^{1/2}y^{-1/2}|| \leqslant 1$  and thus  $||x^{1/2}y^{-1}x^{1/2}|| \leqslant 1$  which implies that  $||x^{1/2}y^{-1}x^{1/2}|| \leqslant 1$ . By 1.3.5  $y^{-1} \leqslant x^{-1/2}1x^{-1/2} = x^{-1}$ .

1.3.7. We say that a continuous real function f on an interval in R is operator monotone (increasing) if  $x \le y$  implies  $f(x) \le f(y)$  whenever the spectra of x and y belong to the interval of definition for f.

For each  $\alpha > 0$  define  $f_{\alpha}$  on  $]-1/\alpha, \infty[$  by

$$f_{\alpha}(t) = (1 + \alpha t)^{-1}t = [1 - (1 + \alpha t)^{-1}]/\alpha.$$

Since the process of taking inverses is operator monotone decreasing by 1.3.6, it is easy to see that  $f_{\alpha}$  is operator monotone increasing on  $]-1/\alpha$ ,  $\infty[$ . The family of functions  $\{f_{\alpha}\}$  will be used repeatedly in the sequel. Note that  $f_{\alpha}(t) < \min\{t, 1/\alpha\}$  and that  $\lim_{t \to 0} f_{\alpha}(t) = t$  uniformly on compact subsets of  $\mathbf{R}$  when  $\alpha \to 0$ . Moreover,  $f_{\alpha} \ge f_{\beta}$  when  $\alpha \le \beta$  and  $f_{\alpha} \circ f_{\beta} = f_{\alpha+\beta}$  on  $]-(\alpha+\beta)^{-1}$ ,  $\infty[$ . Finally, if t>0 then  $\lim_{t \to 0} \alpha f_{\alpha}(t) = 1$  uniformly on compact subsets of  $]0, \infty[$  when  $\alpha \to \infty$ .

1.3.8. PROPOSITION. If  $0 < \beta \le 1$  the function  $t \to t^{\beta}$  is operator monotone on  $R_+$ .

*Proof.* If  $0 \le x \le y$  then  $f_{\alpha}(x) \le f_{\alpha}(y)$  with  $f_{\alpha}$  as in 1.3.7. Now

$$\int_{0}^{\infty} f_{\alpha}(t)\alpha^{-\beta} d\alpha = \int_{0}^{\infty} (1 + \alpha t)^{-1} t\alpha^{-\beta} d\alpha$$

$$= \int_{0}^{\infty} (1 + \alpha)^{-1} t\alpha^{-\beta} t^{\beta} t^{-1} d\alpha = t^{\beta} \int_{0}^{\infty} (1 + \alpha)^{-1} \alpha^{-\beta} d\alpha = \gamma t^{\beta}$$

with  $\gamma$  in  $\mathbf{R}_+$ . For all t in [0, ||y||] and  $\varepsilon > 0$  there is therefore a large n and an equidistant division  $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m = n$  of the interval [0, n] such that

$$|t^{\beta}-(\gamma m)^{-1}n\sum_{k=1}^{m}f_{\alpha_{k}}(t)\alpha_{k}^{-\beta}|<\varepsilon.$$

It follows that  $y^{\beta} - x^{\beta} \ge -2\varepsilon$ , and since  $\varepsilon$  is arbitrary  $x^{\beta} \le y^{\beta}$ .

1.3.9. PROPOSITION. If  $0 \le x \le y$  implies  $x^{\beta} \le y^{\beta}$  for some  $\beta > 1$  and all x, y in a  $C^*$ -algebra A, then A is commutative.

**Proof.** By iteration we see that if the exponent  $\beta$  preserves order then so does  $\beta^n$  for every n in N. Using 1.3.8 we see that the exponents which preserve order form a segment of  $R_+$ . It suffices therefore to prove the proposition with  $\beta = 2$ .

Take x, y in  $A_+$  and  $\varepsilon > 0$ . Then  $x \le x + \varepsilon y$  whence

$$x^2 \le (x + \varepsilon y)^2 = x^2 + \varepsilon (xy + yx) + \varepsilon^2 y^2$$
.

This gives  $0 \le xy + yx + \varepsilon y^2$  for any  $\varepsilon > 0$ , thus

$$(*) xy + yx \geqslant 0.$$

Set xy = a + ib with a and b in  $A_{sa}$ . Clearly  $a \ge 0$ . But (\*) is valid for any product of positive elements and

$$(**) x(yxy) = a^2 - b^2 + i(ab + ba)$$

from which we conclude that  $a^2 - b^2 \ge 0$ .

The set of numbers  $\alpha \ge 1$  such that  $\alpha b^2 \le a^2$  for all x and y in  $A_+$  with xy = a + ib is therefore non-empty. The set E is also closed, so if it was bounded it would have a largest element, say  $\lambda$ . Thus if x, y belongs to  $A_+$  and xy = a + ib then  $a^2 - \lambda b^2 \ge 0$  and therefore by (\*)

$$(***) 0 \le b^2(a^2 - \lambda b^2) + (a^2 - \lambda b^2)b^2 = b^2a^2 + a^2b^2 - 2\lambda b^4.$$

From (\*\*) we now have

$$\lambda(ab+ba)^2 \leqslant (a^2-b^2)^2,$$

that is

$$\lambda[ab^2a + ba^2b + a(bab) + (bab)a] \le a^4 + b^4 - a^2b^2 - b^2a^2.$$

On the left-hand side we have  $a(bab) + (bab)a \ge 0$  by (\*) and  $ba^2b \ge \lambda b^4$  by assumption and finally,  $ab^2a \ge 0$ . Using this, and inserting (\*\*\*) on the right-hand side we get

$$\lambda^2 b^4 \leqslant a^4 + (1 - 2\lambda)b^4,$$

that is

$$(\lambda^2 + 2\lambda - 1)b^4 \le a^4.$$

By 1.3.8 this implies that  $(\lambda^2 + 2\lambda - 1)^{1/2}b^2 \le a^2$  for all a and b with a + ib = xy and x, y in  $A_+$ . But then  $(\lambda^2 + 2\lambda - 1)^{1/2} \in E$  in contradiction with our choice of  $\lambda$  as the largest element. It follows that E is unbounded, whence  $\alpha b^2 \le a^2$  for all  $\alpha \ge 1$ , i.e. b = 0, and A is commutative.

1.3.10. We say that a continuous real function f on an interval in R is operator convex if for any two operators x, y with spectrum in this interval and any  $\lambda$  in [0,1]

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$
.

We say that f is operator concave if -f is operator convex.

1.3.11. PROPOSITION. The functions  $f_{\alpha}$ ,  $\alpha \ge 0$ , the functions  $t \to t^{\beta}$ ,  $0 < \beta \le 1$ , and the functions  $t \to \log(\varepsilon + t)$ ,  $\varepsilon > 0$ , are all operator concave on  $R_+$ .

*Proof.* If a is a positive invertible operator then from spectral theory we have for each  $\lambda$  in [0,1]

$$[\lambda + (1-\lambda)a]^{-1} \leq \lambda + (1-\lambda)a^{-1}.$$

If x and y are positive invertible operators put  $a = x^{-1/2}yx^{-1/2}$ . Multiplying the inequality above with  $x^{-1/2}$  from both sides, we get by 1.3.5

$$[\lambda x + (1 - \lambda)y]^{-1} \le \lambda x^{-1} + (1 - \lambda)y^{-1},$$

which shows that the function  $t \to t^{-1}$  is operator convex on  $]0, \infty[$ . It follows immediately from the formula in 1.3.7 that the functions  $f_{\alpha}$ ,  $\alpha \ge 0$ , are operator concave on  $\mathbf{R}_{+}$ .

Since operator concavity like operator monotonicity is preserved under limits (uniformly on compact subsets) and under convex combinations we see exactly as in the proof of 1.3.8 that the functions  $t \to t^{\beta}$ ,  $0 < \beta \le 1$ , are operator concave. Finally, for each  $\varepsilon > 0$  put

$$g_{\alpha}(t) = (\alpha + 1)^{-1}(\alpha + t + \varepsilon)^{-1}(t + \varepsilon - 1) = (\alpha + 1)^{-1} - (\alpha + t + \varepsilon)^{-1}.$$

The last expression shows that  $g_{\alpha}$  is operator concave for  $\alpha \ge 0$ . The first expression shows that for each  $t \ge 0$  the function  $\alpha \to g_{\alpha}(t)$  is integrable. An elementary calculation yields

$$\int_{0}^{\infty} g_{\alpha}(t) d\alpha = \log(\varepsilon + t) \text{ for observation properties}$$

and consequently the functions  $t \to \log(\varepsilon + t)$ ,  $\varepsilon > 0$  are operator concave. Incidentally, the argument also shows that the functions are operator monotone.

1.3.12. Notes and remarks. The result in 1.3.3, due to Kelley and Vaught [148], shows that the extra axiom.  $(x*x \ge 0)$  in the original definition of a C\*-algebra (see 1.1.14) was redundant, as Gelfand and Naimark also suspected.

Operator monotone functions were characterized by Löwner [163] as being those continuous functions  $f: I \to R$  which admit a holomorphic extension  $\tilde{f}$  o the upper half plane  $C_+ = \{ \operatorname{Im} z > 0 \}$  such that  $\tilde{f}(C_+) \subset C_+$ . It follows by a slight variation of Herglotz's formula that each function f which is operator monotone on  $R_+$  has a unique representation  $f(t) = \int_0^\infty f_a(t) \, \mathrm{d}\mu(\alpha)$  for some positive measure  $\mu$  on  $R_+$ . The result in 1.3.9 is found in [179]. Operator convexity and monotonicity is treated in [16].