Contact problems in the classical theory of elasticity

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PREFACE

Problems concerning the contact between elastic bodies have provided a challenge to applied mathematicians ever since the work of Heinrich Hertz in the 1860's. This book is concerned with idealized versions of these problems as they appear in the classical theory of elasticity. This theory is widely regarded as one of the most beautiful and challenging fields of classical linear mechanics. During the last hundred years it has been the cradle in which a number of powerful mathematical methods have grown. Amongst these are the complex variable method conceived by Kolosoff in the early years of the twentieth century and brought to maturity by Muskhelishvili and his coworkers in Tbilisi. Muskhelishvili's great contribution was the introduction of a systematic, direct method to replace the guesswork by which the complex potentials had previously been chosen. Muskhelishvili's work is described briefly in Chapters III and IV of this present book. It is hoped that these chapters will inspire the reader to study Muskhelishvili's treatises (1953a), (1953b) on elasticity and singular integral equations.

The second powerful mathematical tool which has been sharpened by its use in elasticity theory is the integral transform. Integral transforms were developed in piecemeal fashion during the nineteenth century but, in the author's opinion, it was the seminal work of I.N. Sneddon in his Fourier Transforms (1951) that showed how they could be used for the actual solution of the difficult boundary value problems of elasticity theory. In particular it was he who rescued Ida Busbridge's work on dual integral equations from oblivion and who reworded it to make it accessible to applied mathematicians. Through his writings, his students and his coworkers, his influence can be traced in much of the modern research on classical contact problems.

In writing this book I have been continually aware of the gaps in its coverage. As if the limitation to classical problems were not enough, I have excluded almost entirely a discussion of uniqueness and existence of solutions in elasticity theory, preferring to concentrate on mathematical methods which can be used to find a solution (hopefully the unique one) if one exists. This omission may be viewed as just one aspect of a larger omission of mathematical rigour. Applied mathematicians with pure mathematical inclinations may be inclined to dismiss much of the analysis as *formal*, but I believe that they will find that the analysis may be hedged about with provisos that will make it acceptable in a pure mathematical sense and moreover leave it essentially the same.

My aim in writing the book has been to present a fairly complete account of the subject which may be read without extensive reference to other books. Thus in the book there will be found accounts of the theory of elasticity (Chapter I), complex variable theory (Chapter III), integral transforms (Chapter V) and elliptic functions (Chapter XII). All these accounts start from fundamentals and aim to provide the reader with all the results that he will need to understand the accompanying analysis of contact problems. On the other hand, it is expected that the reader will consult the original papers which provide the details and extensions of the theory described in the text. These papers are listed in the bibliography. This list attempts to give a fairly complete coverage of the Western literature, and of the Soviet literature which has appeared in the more easily accessible Soviet journals. Many of the papers which are cited in the books on Contact Problems in Elasticity Theory by Shtaerman (1949) and Galin (1961) do not fall into this category. I did not think it was worthwhile to include them just to make the list complete.

The reader may supplement his reading with solution of the Examples which are to be found at the end of most sections. Many of these examples rely on particular integral transforms and the reader would do well to acquire handbooks such as Abramowitz and Stegun (1972) and Gradshteyn and Ryzhik (1965).

Preface

The whole of the book was read in manuscript by A.H. England of Nottingham University and L.M. Keer of Northwestern University. They eliminated many of the rough spots; I am responsible for those that remain. Early versions of the manuscript were typed by Jan Daum, Debbie Mustin and Jane Skinner. The final copy was produced in the office of the Solid Mechanics Division of the University of Waterloo. My heartfelt thanks are due to Solid Mechanics Division's Publication Officer D.E. Grierson, Assistant Pam McCuaig, typists Cynthia Jones and Linda Strouth. Louise Adamson cheerfully undertook the Herculean tasks of proofreading successive versions of the manuscript, checking the literature entries and preparing the index.

The book was made possible through a research grant from the National Science and Engineering Council of Canada.

I would be most grateful to be notified of any errors in the text and Examples, and any omissions in the bibliography.

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Waterloo, Ontario, Canada
March, 1980

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Abramian, B.L. and Arutiunian, N.Kh., On some contact problems for composite half-spaces.
80,Vol.31(1967)1001-1008[1007-1013]. 570

appears in Russian in Journal no. 80, namely *Prikladnaia matematika i mekhanika,* pages 1001-1008. The English translation appears in *Journal of Applied Mathematics and Mechanics,* pages 1007-1013. The article is cited on page 570 of this book.

CHAPTER I - THE CLASSICAL THEORY OF ELASTICITY

> "Where shall I begin, please your Majesty"? he asked. "Begin at the beginning" the King said, very gravely, "and go on till you come to the end: then stop."
>
> - *Alice in Wonderland*

1.1 *Introduction*

This book is concerned with a certain class of problems, so-called contact problems, which appear within the theory of elasticity. In this chapter we present an account of those parts of the infinitesimal theory of elasticity which are necessary for the understanding of the problems which will appear later. For a more comprehensive account of this theory the reader should refer to the classical treatises. Love (1927) gives an exhaustive account with numerous historical references which has formed the basis for much of the literature on elasticity in the West. Muskhelishvili's treatise (1953a) is the basis of much of the Russian work, particularly that using complex variable methods. Lur'e (1964) gives a detailed presentation with copious bibliographical notes covering the Russian literature up to the 1950's, and is one of the few treatises dealing explicitly with the effective solutions of 3-dimensional problems. The Handbuch der Physik articles by Sneddon and Berry (1958) and the more recent one by Gurtin (1972) are authoritative, Timoshenko and Goodier (1951) give a presentation of the theory more suited to the engineer, which may prove a useful starting point for those new to the field. Sokolnikoff (1946) emphasizes the tensorial character of the equations of elasticity and is particularly valuable for the chapters on general variational theorems. Green and Zerna (1954) also employ a tensorial approach to 3-dimensional problems, but develop the complex variable methods for plane problems.

The literature on contact problems has been reviewed by a number of authors. Shtaerman (1949), Galin (1961), Ufliand (1965), Rvachev (1967) and Abramian (1971) are concerned mostly with the Soviet literature. Kalker (1975) emphasizes non-classical and rolling contact phenomena which are not discussed in this book. Kalker (1977a) reviewed both classical and non-classical contact phenomena. Hetenyi (1946), (1964), Vlasov and Leont'ev (1966), Popov (1972b), Gorbunov-Posadov and Malikova (1973), and Iyer (1973) are concerned with beams and plates on elastic foundations. References to Hertzian contact may be found in Morton and Close (1922), Way (1940), Timoshenko and Goodier (1951, p. 372f), Dinnik (1952) and Krolevets (1966).

No attempt has been made to include in the bibliography all the references listed in the works cited above. This book is concerned primarily with the study of contact problems as a topic of applied mathematics. We have therefore ignored many researches treating important practical contact problems when we have felt that they are concerned primarily with computing approximate results, rather than illustrating a new mathematical method. (Unfortunately, it cannot be claimed that the cited works all illustrate new mathematical principles. Too many papers have been written in which known, even if only recently discovered, mathematical methods are applied to a problem which differs from previously solved problems only in some small detail.) On the other hand we have paid little attention to pure mathematical questions such as the existence and uniqueness of solutions to contact problems. In general these questions have not been fully answered, but see e.g. Turteltaub and Sternberg (1967), Kalker (1968).

1.2 *The analysis of strain*

In accordance with our limited objective stated in the previous section we shall endeavour to present an analysis of strain which introduces as few complications as possible.

Consider a *continuum* occupying a certain region in 3-dimensional space, and suppose that, by some means, the continuum is *deformed* so that its points undergo *displacement*. Let the displacement of the point P which had coordinates (x,y,z) referred to axes Ox,Oy,Oz fixed in space be $\vec{d} = (u,v,w)$. This is often denoted $\vec{u} = (u,v,w)$. After deformation P therefore occupies the position $P' \equiv (x+u,\ y+v,\ z+w)$, as shown in Figure 1.2.1. If we assume that u,v,w are differentiable functions of the coordinates

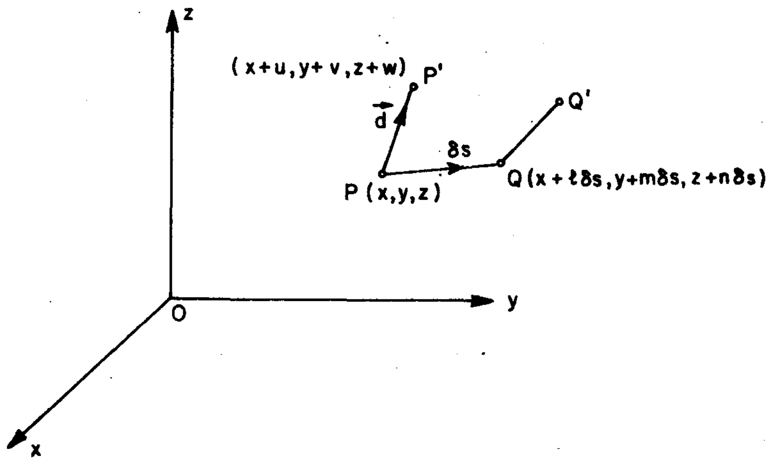

Figure 1.2.1 - The points P,Q *are displaced to* P',Q'.

x,y,z, then the displacement of a neighbouring point $Q(x+\ell\delta s,\ y+m\delta s,\ z+n\delta s)$ distant δs from P along a line with direction cosines (ℓ,m,n) will be

$$u + \delta u = u(x+\ell\delta s,\ y+m\delta s,\ z+n\delta s) = u(x,y,z) + \\ + \delta s\left\{\ell\frac{\partial u}{\partial x} + m\frac{\partial u}{\partial y} + n\frac{\partial u}{\partial z} + \dots\right\}, \tag{1.2.1}$$

$$v + \delta v = v(x+\ell\delta s,\ y+m\delta s,\ z+n\delta s) = v(x,y,z) + \\ + \delta s\left\{\ell\frac{\partial v}{\partial x} + m\frac{\partial v}{\partial y} + n\frac{\partial v}{\partial z} + \dots\right\}, \tag{1.2.2}$$

$$w + \delta w = w(x+\ell\delta s,\ y+m\delta s,\ z+n\delta s) = w(x,y,z) + $$
$$+ \delta s\left\{\ell\,\frac{\partial w}{\partial x} + m\,\frac{\partial w}{\partial y} + n\,\frac{\partial w}{\partial z} + \ldots\right\}, \qquad (1.2.3)$$

where all the derivatives are evaluated at (x,y,z). Provided that δs is sufficiently small, the higher terms in (1.2.1)-(1.2.3) which involve powers of δs may be neglected.

We now restrict our analysis to *infinitesimal* deformations by assuming that the products of derivatives of u,v,w may be neglected in comparison with the linear terms. The distance between the displaced points P',Q' will be $\delta s + \Delta s$, where

$$(\delta s + \Delta s)^2 = (\delta s)^2\left[\left\{\ell\left(1 + \frac{\partial u}{\partial x}\right) + m\,\frac{\partial u}{\partial y} + n\,\frac{\partial u}{\partial z}\right\}^2 + \right.$$
$$+ \left\{\ell\,\frac{\partial v}{\partial x} + m\left(1 + \frac{\partial v}{\partial y}\right) + n\,\frac{\partial v}{\partial z}\right\}^2 +$$
$$\left. + \left\{\ell\,\frac{\partial w}{\partial x} + m\,\frac{\partial w}{\partial y} + n\left(1 + \frac{\partial w}{\partial z}\right)\right\}^2\right].$$

Under the restriction to infinitesimal strain the proportional increase in the length of the line PQ due to the deformation is

$$\frac{\Delta s}{\delta s} = \ell^2\,\frac{\partial u}{\partial x} + m^2\,\frac{\partial u}{\partial y} + n^2\,\frac{\partial w}{\partial z} + mn\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) +$$
$$+ n\ell\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \ell m\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

The *coefficients* in this expression are called the *components of strain* and are denoted by

$$e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{yy} = \frac{\partial v}{\partial y}, \qquad e_{zz} = \frac{\partial w}{\partial z}, \qquad (1.2.4)$$
$$e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),\ e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right),\ e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \qquad (1.2.5)$$

Thus

$$\frac{\Delta s}{\delta s} = \ell^2 e_{xx} + m^2 e_{yy} + n^2 e_{zz} + 2mn e_{yz} +$$
$$+ 2n\ell e_{zx} + 2\ell m e_{xy}. \qquad (1.2.6)$$

Note that Love (1927) omits the factor 1/2 in the *shear strains* e_{yz}, e_{zx}, e_{xy}; many authors omit the second suffix in the *normal strains* e_{xx}, e_{yy}, e_{zz}: Timoshenko and Goodier (1951) denote them by $\varepsilon_x, \varepsilon_y, \varepsilon_z$. Note that the shear strains can be treated as *symmetric* quantities: i.e. we introduce e_{zy}, e_{xz}, e_{yx}, where

$$e_{zy} = e_{yz}, \quad e_{xz} = e_{zx}, \quad e_{yx} = e_{xy}. \tag{1.2.7}$$

If the displacement of the body is due to a combination of *rigid body displacement*, in which each point receives the same displacement, and an infinitesimal *rigid body rotation* $(\omega_1, \omega_2, \omega_3)$, in which $\omega_1, \omega_2, \omega_3$ represent the angles of rotation about the three axes Ox, Oy, Oz respectively, then equations (1.2.1)-(1.2.3) become

$$u + \delta u = u - \omega_3 m\delta s + \omega_2 n\delta s,$$

$$v + \delta v = v - \omega_1 n\delta s + \omega_3 \ell\delta s,$$

$$w + \delta w = w - \omega_2 \ell\delta s + \omega_1 m\delta s.$$

Under such a deformation all the components of strain are zero since, for example,

$$\frac{\partial u}{\partial y} = -\omega_3 = -\frac{\partial v}{\partial x}.$$

This means that if we introduce three *components of rotation*, defined by

$$\left.\begin{aligned} \omega_{yz} &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \\ \omega_{zx} &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \\ \omega_{xy} &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \end{aligned}\right\} \tag{1.2.8}$$

then equations (1.2.1)-(1.2.3) may be written

$$u + \delta u = u + \delta s\{\ell e_{xx} + m e_{xy} + n e_{xz} - m\omega_{xy} + n\omega_{zx}\},$$

$$v + \delta v = v + \delta s\{\ell e_{yx} + m e_{yy} + n e_{yz} - n\omega_{yz} + \ell\omega_{xy}\},$$

$$w + \delta w = w + \delta s\{\ell e_{zx} + m e_{zy} + n e_{zz} - \ell\omega_{zx} + m\omega_{yz}\}.$$

The strains may be given physical interpretations. By putting $m = 0 = n$ we see that e_{xx} represents the proportional change of length of the line joining $P(x,y,z)$ to $Q(x+\ell\delta s, y,z)$. Again, consider two points $Q(x+\ell\delta s_1, y,z)$ and $R(x, y+m\delta s_2, z)$ neighbouring P as in Figure 1.2.2; initially PQ and PR are orthogonal.

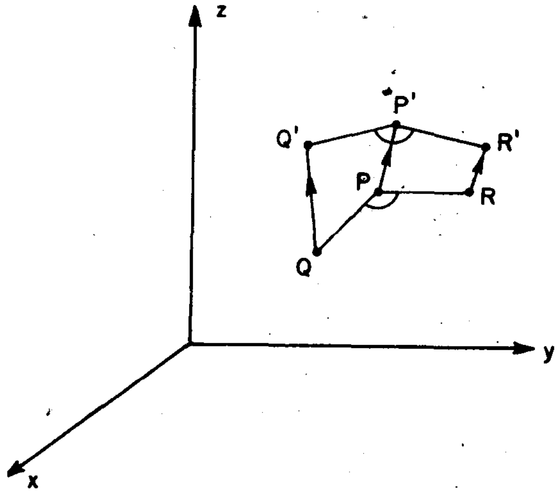

Figure 1.2.2 - The right-angle QPR *is changed to* Q'P'R'.

Under deformation P goes to $P'(x+u, y+v, z+w)$ while Q,R go respectively to Q',R' with coordinates

$$(x+u+\ell\delta s_1(1+e_{xx}),\ y+v+\ell\delta s_1(e_{yx}+\omega_{xy}),\ z+w+\ell\delta s_1(e_{zx}-\omega_{zx}))$$

$$(x+u+m\delta s_2(e_{xy}-\omega_{xy}),\ y+v+m\delta s_2(1+e_{yy}),\ z+w+n\delta s_2(e_{zy}-\omega_{yz})).$$

The cosine of the angle $(\pi/2-\alpha)$ between P'Q' and P'R' is given to the first order by

$$\cos(\pi/2-\alpha) = \sin\alpha = \alpha = 1(e_{xy}-\omega_{xy}) + 1(e_{yx}+\omega_{xy}) = 2e_{xy}.$$

Thus $2e_{xy}$ represents the decrease in the angle between the two lines PQ,PR initially at right angles.

The strain components $e_{xx}, e_{xy}, e_{xz}, e_{yx}, e_{yy}, e_{yz}, e_{zx}, e_{zy}, e_{zz}$ form the 9 components of a tensor of rank 2, (see e.g. Sokolnikoff