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H.O.Cordes

拟微分算子技巧

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To my 6 children, Stefan and Susan

Sabine and Art, Eva and Sam

P R E F A C E

It is generally well known that the Fourier-Laplace transform converts a linear constant coefficient PDE $P(D)u=f$ on $\mathbb{R}^n$ to an equation $P(\xi)u^{\wedge}(\xi)=f^{\wedge}(\xi)$, for the transforms $u^{\wedge}$, $f^{\wedge}$ of u and f, so that solving $P(D)u=f$ just amounts to division by the polynomial $P(\xi)$. The practical application was suspect, and ill understood, however, until theory of distributions provided a basis for a logically consistent theory. Thereafter it became the Fourier-Laplace method for solving initial-boundary problems for standard PDE. We recall these facts in some detail in sec's 1-4 of ch.0.

The technique of pseudodifferential operator extends the Fourier-Laplace method to cover PDE with variable coefficients, and to apply to more general compact and noncompact domains or manifolds with boundary. Concepts remain simple, but, as a rule, integrals are divergent and infinite sums do not converge, forcing lengthy, often endlessly repetitive, discussions of 'finite parts' (a type of divergent oscillatory integral existing as distribution integral) and asymptotic sums (modulo order $-\infty$).

Of course, pseudodifferential operators (abbreviated ψdo's) are (generate) abstract linear operators between Hilbert or Banach spaces, and our results amount to 'well-posedness' (or normal solvability) of certain such abstract linear operators. Accordingly both, the Fourier-Laplace method and theory of ψdo's, must be seen in the context of modern operator theory.

To this author it always was most fascinating that the same type of results (as offered by elliptic theory of ψdo;'s) may be obtained by studying certain examples of Banach algebras of linear operators. The symbol of a ψdo has its abstract meaning as Gelfand function of the coset modulo compact operators of the abstract operator in the algebra.

On the other hand, hyperbolic theory, generally dealing with a group exp(Kt) (or an evolution operator U(t)) also has its manifestation with respect to such operator algebras: conjugation with exp(Kt) amounts to an automorphism of the operator algebra, and of the quotient algebra. It generates a flow in the symbol space essentially the characteristic flow of singularities. In [C_1], [C_2] we were going into details discussing this abstract approach.

We believe to have demonstrated that ψdo's are not necessary to understand these fact. But the technique of ψdo's, in spite of its endless formalisms (as a rule integrals are always 'distribution integrals', and infinite series are asymptotically convergent, not convergent), still provides a strongly simplifying principle, once the technique is mastered. Thus our present discussion of this technique may be justified.

On the other hand, our hyperbolic discussions focus on invariance of ψdo-algebras under conjugation with evolution operators, and do not touch the type of oscillatory integral and further discussions needed to reveal the structure of such evolution operators as Fourier integral operators. In terms of Quantum mechanics we prefer the Heisenberg representation, not the Schroedinger representation.

In particular this leads us into a discussion of the Dirac equation and its invariant algebra, in chapter X. We propose it as algebra of observables.

The basis for this volume is (i) a set of notes of lectures given at Berkeley in 1974-80 (chapters I-IV) published as preprint at U. of Bonn, and (ii) a set of notes on a seminar given in 1984 also at Berkeley (chapters VI-IX). The first covers elliptic (and parabolic) theory, the second hyperbolic theory. One might say that we have tried an old fashiened PDE lecture in modern style.

In our experience a newcomer will have to reinvent the theory before he can feel at home with it. Accordingly, we did not try to push generality to its limits. Rather, we tend to focus on the simplest nontrivial case, leaving generalizations to the reader. In that respect, perhaps we should mention the problems (partly of research level) in chapters I-IV, pointing to manifolds with conical tips or cylindrical ends, where the 'Fredholm-significant symbol' becomes operator-valued.

The material has been with the author for a long time, and was subject of many discussions with students and collaborators. Especially we are indebted to R. McOwen, A.Erkip, H. Sohrab, E. Schrohe, in chronological order. We are grateful to Cambridge University Press for its patience, waiting for the manuscript.

Berkeley, November 1993 Heinz O. Cordes

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Chapter 0. INTRODUCTORY DISCUSSIONS.

In the present introductory chapter we give comprehensive discussions of a variety of nonrelated topics. All of these bear on the concept of pseudo-differential operator, at least in the author's mind. Some are only there to make studying ψdo's appear a natural thing, reflecting the author's inhibitions to think along these lines.

In sec.1 we discuss the elementary facts of the Fourier transform, in sec.'s 2 and 3 we develop Fourier-Laplace transforms of temperate and nontemperate distributions. In sec.4 we discuss the Fourier-Laplace method of solving initial-value problems and free space problems of constant coefficient partial differential equations. Sec.5 discusses another problem in PDE, showing how the solving of an abstract operator equation together with results on hypo-ellipticity and "boundary-hypo-ellipticity" can lead to existence proofs for classical solutions of initial-boundary problems. Sec.6 is concerned with the operator e^{Lt}, for a first order differential expression L. Sec.'s 7 and 8 deal with the concept of characteristics of a linear differential expression and learning how to solve a first order PDE. Sec.9 gives a mini-introduction to Lie groups, focusing on the mutual relationship between Lie groups and Lie algebras. (Note the relation to ψdo's discussed in ch.8).

We should expect the reader to glance over ch.0 and use it to have certain prerequisites handy, or to get oriented in the serious reading of later chapters.

0. Some special notations.

The following notations, abbreviations, and conventions will be used throughout this book.

(a) $\kappa_n = (2\pi)^{-n/2}$, $đx = \kappa_n dx_1 dx_2 \ldots dx_n = \kappa_n dx$.

(b) $\langle x\rangle = (1+|x|^2)^{1/2}$, $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, etc.

(c) Derivatives are written in various ways, at convenience: For $u=u(x)=u(x_1,\dots,x_n)$ we write $u^{(\alpha)}=\partial_x^\alpha u=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\dots u = =\partial^{\alpha_1}/\partial x^{\alpha_1}\dots\partial^{\alpha_n}/\partial x^{\alpha_n}u$. Or, $u_{|x_j}=\partial_{x_j}u$, $u_{|x}$ to denote the n-vector with components $u_{|x_j}$, $\nabla_x^k u$ for the k-dimensional array with components $u_{|x_{i_1}x_{i_2}\dots}$. For a function of $(x,\xi)=(x_1,\dots,x_n,\xi_1,\dots,\xi_n)$ it is often convenient to write $u_{(\beta)}^{(\alpha)}=\partial_\xi^\alpha\partial_x^\beta u$.

(d) A multi-index is an n-tuple of integers $\alpha=(\alpha_1,\dots,\alpha_n)$. We write $|\alpha|=|\alpha_1|+\dots+|\alpha_n|$, $\alpha!=\alpha_1!\dots\alpha_n!$, $\binom{\alpha}{\beta}=\binom{\alpha}{\beta}\dots\binom{\alpha}{\beta}$, $x^\alpha=x_1^{\alpha_1}\dots x_n^{\alpha_n}$, etc., $\mathbb{N}^n$={all multi-indices}.

(e) Some standard spaces: $\mathbb{R}^n$ = n-dimensional Euclidean space $\mathbb{B}^n$=directional compactification of $\mathbb{R}^n$ (one infinite point ∞x added in every direction (of a unit vector x).

(f) Spaces of continuous or differentiable complex-valued functions over a domain or differentiable manifold X (or sometimes only $X=\mathbb{R}^n$): C(X) = continuous functions on X ; CB(X)= bounded continuous functions on X; CO(X)= continuous functions on X vanishing at ∞ ; CS(X) = continuous functions with directional limits; $C_0(X)$ = continuous functions with compact support; $C^k(X)$= functions with derivatives in C, to order k, (incl. $k=\infty$). $CB^\infty(X)$="all derivatives exist and are bounded". The Laurent-Schwartz notations $\mathcal{D}(X)=C_0^\infty(X)$, $\mathcal{E}(X)=C^\infty(X)$ are used. Also $\mathcal{S}=\mathcal{S}(\mathbb{R}^n)$= "rapidly decreasing functions" (All derivatives decay stronger as any power of x). Also, distribution spaces $\mathcal{D}'$, $\mathcal{E}'$, $\mathcal{S}'$.

(g) L^p-spaces: For a measure space X with measure $d\mu$ we write $L^p(X)=L^p(X,d\mu)$={measurable functions u(x) with $|u|^p$ integrable} for $1\le p<\infty$; $L^\infty(X)$={essentially bounded functions}.

(h) Maps between general spaces: C(X,Y) denotes the continuous maps $X\to Y$. Similar for the other symbols under (f), i.e., CB(X,Y) ,.... .

(i) Classes of linear operators (X= Banach space) : $L(X)$ ($K(X)$)= continuous (compact) operators; $GL(X)$ ($U(H)$) = invertible (unitary) operators of $L(X)$ (of $L(H)$, H=Hilbert space); $U_n=U(\mathbb{C}_n)$. For operators $X\to Y$, again, $L(X,Y)$, etc.

(j) The convolution product: For $u,v\in L^1(\mathbb{R}^n)$ we write $w(x) =(u*v)(x)=\kappa_n\int dy\,u(x-y)v(y)$ (Note the factor $\kappa_n=(2\pi)^{-n/2}$).

(k) Special notation: " $X \subset\subset Y$ " means that X is contained in a compact subset of Y .

(l) For technical reason we may write $\lim_{\varepsilon\to 0} a(\varepsilon) = a|_{\varepsilon\to 0}$.

(m) Abbreviations used: ODE (PDE) = ordinary (partial) differential equation (or "expression"). FOLPDE (or folpde)= first order linear partial differential equation (or "expression"); ψdo= pseudodifferential operator.

(n) Integrals need not be existing (proper or improper) Riemann or Lebesgue integrals, unless explicitly stated, but may be distribution integrals By this term we mean that either (i) the integral may be interpreted as value of a distribution at a testing function-the integrand may be a distribution, or (ii) the limit of Riemann sums exists in the sense of weak convergence of a sequence of (temperate) distributions, or (iii) the limit defining an improper Riemann integral exists in the sense of weak convergence, as above, or (iv) the integral may be a 'finite part' (cf. I,4).

(o) Adjoints: For a linear operator A we use 'distribution adjoint' A^{-} and 'Hilbert space adjoint' A^{*}, corresponding to transpose A^{T} and adjoint $\bar{A}^{T}=A^{*}$, in case of a matrix $A=((a_{jk}))$, respectively. For a symbols $a(x,\xi)$, a^{*} (or a^{+}) may denote the symbol of the adjoint ψdo of $a(x,D)$, as specified in each section.

(p) supp u (sing supp u (or s.s.u)) denotes the (singular) support of the distribution u.

1. The Fourier transform; elementary facts.

Let $u \in L^{1}(\mathbb{R}^{n})$ be a complex-valued integrable function. Then we define the Fourier transform $u^{\wedge} = Fu$ of u by the integral

$$u^{\wedge}(x) = \int_{\mathbb{R}^n} d\!\!\!/\xi\, u(\xi) e^{-ix\xi} \quad , \; x \in \mathbb{R}^{n} , \tag{1.1}$$

with $x\xi = x.\xi = \sum_{j=1}^{n} x_j \xi_j$, an existing Lebesgue integral. Clearly,

$$|u^{\wedge}(x)| \le \|u\|_{L_1} = \int_{\mathbb{R}^n} d\!\!\!/x\, |u(x)| \; . \tag{1.2}$$

Note that $u^{\wedge}$ is uniformly continuous over $\mathbb{R}^{n}$: We get

$$\begin{aligned} |u^{\wedge}(x)-u^{\wedge}(y)| &\le 2\int d\!\!\!/\xi\, |\sin(x-y)\xi/2|\,|u(\xi)| \\ &\le N|x-y|\,\|u\|_{L^1} + 2\int_{|\xi|\ge N} d\!\!\!/\xi\, |u(\xi)| \; , \end{aligned} \tag{1.3}$$

where the right hand side is $<\varepsilon$ if N is chosen for $\int_{|\xi|\ge N} < \varepsilon/4$, and then $|x-y| < \varepsilon/(2N\|u\|_{L^1})$. Moreover, we get $u^\wedge \in CO(\mathbb{R}^n)$, i.e., $\lim_{|x|\to\infty} u^\wedge(x)=0$, a fact, known as the Riemann-Lebesgue lemma.

To prove the latter, we reduce it to the case of $u \in C_0^\infty(\mathbb{R}^n)$: The space C_0^∞ is known to be dense in L^1. By (1.1) we get

$$(1.4) \qquad |u^\wedge(x)-v^\wedge(x)| \le \|u-v\|_{L^1} < \varepsilon/2 \ , \text{ as } v \in C_0^\infty \ , \ \|u-v\|_{L^1}<\varepsilon/2.$$

Hence $\lim_{|x|\to\infty} v^\wedge(x)=0$ implies $|u^\wedge| \le |u^\wedge - v^\wedge| + |v^\wedge| < \varepsilon$ whenever x is chosen according to $|v^\wedge| < \varepsilon/2$.

But for $v \in C_0^\infty$ the Fourier integral extends over a ball $|\xi| \le N$ only, since v=0 outside. We may integrate by parts for

$$(1.5) \quad |x|^2 u^\wedge(x) = -\int \Delta_\xi (e^{-ix\xi}) v(\xi) đ\xi = -\int đ\xi e^{-ix\xi} (\Delta v)(\xi) = -(\Delta v)^\wedge(x) \ ,$$

with the Laplace differential operator $\Delta_\xi = \sum_{j=1}^n \partial_{\xi_j}{}^2$. Clearly we have $\Delta v \in C_0^\infty \subset L^1$ as well, whence (1.1) applies to Δv, for

$$(1.6) \qquad |v^\wedge(x)| \le \|\Delta v\|_{L^1}/|x|^2 \to 0 \ , \text{ as } |x| \to \infty \ ,$$

completing the proof.

The above partial integration describes a general method to be applied frequently in the sequel. (1.6) may be derived under the weaker assumptions that $v \in C^2$, and that all derivatives $v^{(\alpha)}$, $|\alpha| \le 2$, are in L^1 (cf. pbm.5). On the other hand, there are simple examples of $u \notin L^1$ such that $u^\wedge$ does not decay as rapidly as (1.6) indicates. In particular, $u \in L^1$ exists with $u^\wedge \notin L^1$ (cf.pbm.4).

This matter becomes important if we think of inverting the linear operator $F: L^1 \to CO$ defined by (1.1), because formally an inverse seems to be given by almost the same integral. Indeed, define the (complex) conjugate Fourier transform $\overline{F}: L^1 \to CO$ by $\overline{F}u = \overline{(F\overline{u})}$, or, $u^\vee = \overline{F}u$, where

$$(1.7) \qquad u^\vee(x) = \int đ\xi e^{ix\xi} u(\xi) \quad , \ u \in L^1(\mathbb{R}^n) \ .$$

Then, in essence, it will be seen that $\overline{F}$ is the inverse of the operator F. More precisely we will have to restrict F to a (dense) subspace of L^1, for this result. Or else, the definition of the operator $\overline{F}$ must be extended to certain non-integrable functions, for which existence of the Lebesgue integral (1.7) cannot be expected. Both things will be done, eventually.

It turns out that F induces a unitary operator of the Hilbert space $L^2(\mathbb{R}^n)$: We have <u>Parseval's relation</u>:

$$(1.8)\quad \int_{\mathbb{R}^n} đx|u^{\wedge}(x)|^2 = \int_{\mathbb{R}^n} đx|u(x)|^2 \ , \text{ for all } u \in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n) \ .$$

Formula (1.8) is easier to prove as the <u>Fourier inversion formula</u>, asserting $u^{\wedge\vee}=u^{\vee\wedge}=u$ for certain u: We may write

$$(1.9)\quad \int_{Q_N} đx\bar{u}^{\wedge}(x)v^{\wedge}(x) = \int đ\xi đ\eta\bar{u}(\xi)v(\eta)\Pi_{j=1}^n \int_{-N}^{N} e^{ix_j(\xi_j-\eta_j)}đx_j \ ,$$

assuming that $u,v\in L^1(\mathbb{R}^n)$, with the 'cube' $Q_N=\{|x_j|\le N, j=1,\dots,N\}$, some integer N>0. Indeed, the interchange of integrals leading to (1.9) is legal, since the integrand is $L^1(Q_N\times\mathbb{R}^n\times\mathbb{R}^n)$.

Note that $\int_{-N}^{N} e^{ist}dt = 2\frac{\sin sN}{s}$, $s\neq 0$, $= 2N$, $s=0$, allowing evaluation of the inner integrals at right of (1.9). With $\int đ\xi đ\eta = \int đ\xi\int đ\eta$, and $\eta = \xi - \zeta/N$, $đ\eta = N^{-n}đ\zeta$, (1.9) assumes the form

$$(1.10)\quad \int_{Q_N} đx\bar{u}^{\wedge}(x)v^{\wedge}(x) = \int đ\xi\bar{u}(\xi)\int đ\zeta v(\xi-\zeta/N)\Pi_{j=1}^n\varphi(\zeta_j) \ ,$$

where $\varphi(t) = (2/\pi)^{1/2}\frac{\sin t}{t}$, $t\neq 0$, continuously extended into t=0.

For $v\in C(R^n)$, as $N\to\infty$, the function $v(\xi-\zeta/N)$ will converge to $v(\xi)$, independent of ζ. Thus one expects the inner integral at right of (1.10) to converge to $v(\xi)\int\Pi_{j=1}^n\varphi(\zeta_j)đ\zeta_j = v(\xi)$, since

$$(1.11)\quad \int_0^\infty \sin t \ dt/t = \pi/2 \ .$$

Legalization of this argument will confirm Parseval's relation, since the right hand converges to the right hand side of (1.8), as $N\to\infty$. With $u\in L^1$ and $v\in C_0^\infty$ (setting $\varphi_n(\zeta)=\Pi\varphi(\zeta_j)$) write

$$(1.12)\quad \int đ\xi\bar{u}(\xi)\int đ\zeta\varphi_n(\zeta)(v(\xi-\zeta/N)-v(\xi)) = \int_{Q_N} đx\bar{u}^{\wedge}v^{\wedge} - \int_{R^n} đx\bar{u}v \ .$$

To show that the inner integral at left goes to 0 as $N\to\infty$ it is more skilful to use the integration variable $\theta=\zeta/N$, $d\zeta=N^n d\theta$. For

$$n=1 \ , \int \sin N\theta \ (v(\xi-\theta)-v(\xi))d\theta/\theta = \int_{|\theta|\le\delta} + \int_{|\theta|\ge\delta} = I_0 + I_\infty \ .$$

Here we get (with $w(\theta) = (v(\xi-\theta)-v(\xi))/\theta$)

$$|I_0| \le\delta\|v'\|_{L^\infty}, \ I_\infty=\tfrac{1}{N}((w(\theta)\cos(N\theta)|_{\theta=-\delta}^{\theta=\delta} + \int_{|\theta|\ge\delta}\cos(N\theta)w_{|\theta}(\theta)d\theta).$$

The latter gives $I_\infty \le \frac{c}{N\delta}(\|v\|_{L^\infty}+\|v'\|_{L^\infty})$, with a constant *c*, *only* depending on the volume of supp v, i.e., it is fixed after fixing v. The estimates imply the inner integral *to go to* 0, uniformly as $x \in \mathbb{R}^n$. For $u \in L^1$ the Lebesgue theorem then implies the left hand side of (1.12) to tend to 0, as $N \to \infty$, for each fixed $v \in C_0^\infty$.

For general n the proof is a bit less transparent, but remains the same: Split the inner integral into a sum of integrals over a small neighbourhood of 0 and its complement. In the first term use differentiability of v; in the second an integration by parts.

We now have a 'polarized' Parseval relation, in the form

$$(1.13) \qquad \int_{R^n} đx \bar{u}^\wedge v^\vee = \int_{R^n} đx \bar{u} v \ , \ \text{for } u \in L^1 \ , \ v \in C_0^\infty \ .$$

For $u \in L^1 \cap L^2$ pick a sequence $u_j \in C_0^\infty$ with $\|u-u_j\|_{L^1} \to 0$, $\|u-u_j\|_{L^2} \to 0$, as is possible. Then, since $u_j - u_l \in C_0^\infty \subset L^2$, (1.13) with $u=v=u_j-v_j$ implies $\|u_j^\wedge - u_l^\wedge\|_{L^2} = \|u_j - u_l\|_{L^2} \to 0$, $j,l \to \infty$. In other words, u_j and $u_j^\wedge$ both converge in L^2. Clearly, $u_j^\wedge \to u^\wedge$. Indeed, initially we showed uniform convergence over $\mathbb{R}^n$, while the L^2-limit $z = \lim u_j^\wedge$ satisfies $(u^\wedge, \varphi) = \int \bar{z} \varphi dx$ for all $\varphi \in C_0^\infty$. This yields $\int (u^\wedge - z) \varphi dx = 0$ for all such φ, hence $\hat{u} = z$ (almost everywhere), since C_0^∞ is dense in L^2. Substituting $u=v=u_j$ in (1.13), letting $j \to \infty$, it follows that (1.8) is valid for all $u \in L^1 \cap L^2$, confirming Parseval's relation.

Clearly (1.13) also holds for all $u,v \in L^1 \cap L^2$. We use it to prove the Fourier inversion Let n=1. For $v \in L^1 \cap L^2$, $u = \chi_{[0,x_0]}$, some $x_0 > 0$ apply (1.13). Confirm by calculation of the integral that

$$(1.14) \qquad (2\pi)^{1/2} u^\wedge(x) = (e^{-ixx_0} - 1)/(-ix) = h_{x_0}(x) \ , \quad x \neq 0 \ ,$$

hence

$$(1.15) \qquad \int_0^{x_0} v(x)dx = \int đx v^\wedge(x) \bar{h}_{x_0}(x) dx \quad .$$

The Fourier inversion formula is a matter of differentiating (1.15) for x_0 under the integral sign, assuming that this is legal Consider the difference quotient:

$$(1.16) \qquad (2\delta)^{-1} \int_{x_0-\delta}^{x_0+\delta} v(x)dx = \int đx v^\wedge(x) e^{ixx_0} \frac{\sin \delta x}{\delta x} \ .$$

Assuming only that v, $v^\wedge$ both are in L^1, it follows indeed that

$$(1.17) \ \lim_{\delta \to 0} (2\delta)^{-n} \int_{Q_{x_0,\delta}} v(x)dx = \int đx v^\wedge(x) e^{ixx_0} = (v^\wedge)^\vee(x_0), \ x_0 \in \mathbb{R}^n.$$

(Actually, our proof works for n=1, $x_0 > 0$ only, but can easily be extended to all x_0, and general n. One must replace the derivative d/dx_0 by a mixed derivative $\partial^n / (\partial x_{0,1} \ldots \partial x_{0,n})$.) Indeed, letting $\delta\to 0$ in (1.17) we obtain (1.15), using that $\sin(\delta x)/(\delta x) \to 1$ uniformly on compact sets, and boundedly on $\mathbb{R}$, as $\delta\to 0$.

If v is continuous at x_0 then clearly the left hand side of (1.17) equals $v(x_0)$, giving the Fourier inversion formula, as it is well known. For n=1, if v has a jump at x_0 then the left hand side of (1.17) equals the mean value $(v(x_0+0)+v(x_0-0))/2$.

Again for n=1 a limit of (1.16), as $\delta\to 0$ exists, if only

$$(1.18) \qquad \lim_{\alpha\to\infty} \int_{-\alpha}^{+\alpha} v^{\wedge}(x)\,đx ,$$

the principal value, exists (cf. pbm.6), without requiring $v^{\wedge} \in L^1$.

We summarize our results thus far:

<u>Proposition 1.1</u>. The Fourier transform $u^{\wedge}$ of (1.1) and its complex conjugate $u^{\vee} = (\bar{u}^{\wedge})^{-}$ are defined for all $u \in L^1(\mathbb{R}^n)$, and we have $u^{\wedge}, u^{\vee} \in CO(\mathbb{R}^n)$. For $u \in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ we have Parseval's relation (1.8). If both $u \in L^1(\mathbb{R}^n)$ and $u^{\wedge} \in L^1(\mathbb{R}^n)$ hold, then we have $u^{\wedge\vee}(x) = u^{\vee\wedge}(x) = u(x)$ for almost all $x \in \mathbb{R}^n$.

It is known that the Banach space $L^1(\mathbb{R}^n)$ is a commutative Banach algebra under the convolution product

$$(1.19) \qquad u*v = w \ , \ w(x) = \int đy\,u(x-y)w(y) = \int đy\,v(x-y)u(y) .$$

Indeed,

$$(1.20) \qquad \|w\|_{L^1} = \int |w(x)|dx \le \kappa_n \int dx \int dy\,|u(x-y)|\,|v(y)| = \kappa_n \|u\|_{L^1}\|v\|_{L^1} .$$

Prop.1.2, below, clarifies the role of the Fourier transform F for this Banach-algebra: F provides the Gelfand homomorphism.

<u>Proposition 1.2</u>. For $u,v \in L^1$ let $w = u*v$. Then we have

$$(1.21) \qquad w^{\wedge}(\xi) = u^{\wedge}(\xi)v^{\wedge}(\xi) \ , \ \xi \in \mathbb{R}^n .$$

<u>Proof</u>. We have

$$w^{\wedge}(\xi) = \int đx\,e^{-ix\xi}\int đy\,u(x-y)v(y) = \int đy\,e^{-iy\xi}\int dx\,u(x-y)e^{-i(x-y)\xi} .$$

The substitution x-y=z, dy=dz thus confirms (1.21), q.e.d.

The importance of the Fourier transform for PDE's hinges on

<u>Proposition 1.3</u>. If $u^{(\beta)} \in L^1$ for all $\beta\le\alpha$ then

$$(1.22) \qquad u^{(\alpha)\wedge}(\xi) = i^{|\alpha|}\xi^{\alpha}u^{\wedge}(\xi) \ , \ \xi \in \mathbb{R}^n .$$