Singular Optimal Control Problems

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PREFACE

The modern theory of optimal control has its beginnings in the pioneering work of men such as Hohmann in Europe and Goddard in the United States. These men, and others like them, dreamt of the day when space travel would become a reality. It is the work of these pioneers which has helped to bring their dreams to partial fulfillment in such a brief period of time. The optimization problems of space flight dynamics such as minimum fuel and minimum time were quickly realized to be problems in the calculus of variations. It was thus that this very old and established branch of Mathematics received yet another new lease of life, a pattern which has been repeated since its birth in the seventeenth century.

Problems of high performance aircraft became important near the end of the Second World War. Maximum range of aircraft for a given quantity of fuel and minimum time to climb were typical optimization problems which arose. These too were clearly problems belonging to the same class as those from space dynamics although complicated by the presence of aerodynamic lift and drag.

Much of the mathematical theory for such problems had already been developed early in the twentieth century by Professor G. A. Bliss and his students at the University of Chicago. In particular, the optimization of aircraft and space vehicle flight paths for which the controls are finite and bounded was analysed by a technique described in a dissertation by F. A. Valentine, a research student of Bliss, in 1937. However, in 1959 L. S. Pontryagin presented his Maximum Principle which consolidated the theory for constrained problems.

Nevertheless, it soon became apparent that the mathematical theory available was not sufficient for certain special control problems in which the Pontryagin Principle yielded no additional information on the stationary control. These problems were described as singular problems and they have arisen in many engineering applications in fields other than Aerospace and also more recently in non-engineering areas such as Economics. In the last ten to twelve years much effort has been put into the development of new theory to deal with these singular problems. First, new necessary conditions were found for singular extremals to be candidate optimal arcs. Secondly, and more recently, new sufficient conditions and necessary and sufficient conditions have been found for such extremals to be optimal.

We feel that the time is now right for the theory of singular problems to be collected together, scattered as it is in numerous different journals, and presented under one cover. This is the purpose of the present volume. We are particularly pleased that this book should take its place alongside so many well-acclaimed texts in Richard Bellman's series which has proved its

worth many times over. Our gratitude goes to all our colleagues and students, past and present, who have stimulated us both in our own researches. In particular, we would like to mention Y. C. Ho, D. Q. Mayne, J. L. Speyer, W. Vandervelde, D. F. Lawden, R. N. A. Plimmer and B. S. Goh. Finally, we acknowledge the help and encouragement received from Academic Press during the preparation of this book and our special thanks go to Mrs. G. M. McEwen and Mrs. M. E. Hughes who typed the manuscript.

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CHAPTER 1

An Historical Survey of Singular Control Problems

1.1 Introduction

It is well known that the fundamental problem of optimal control theory can be formulated as a problem of Bolza, Mayer or Lagrange. These three formulations are quite equivalent to one another (Bliss, 1946). We shall describe the Bolza problem since the accessory minimum problem, and the associated second variation with which we shall be much concerned in this book, appears as such a problem.

The problem of Bolza in optimal control theory is the following. Determine the control function $u(\cdot)$ which minimizes the cost functional

$$J = F[x(t_f),\ t_f] + \int_{t_o}^{t_f} L(x,u,t)dt \tag{1.1.1}$$

where the system equation is

$$\dot{x} = f(x,u,t) \tag{1.1.2}$$

subject to the constraints

$$x(t_o) = x_o \tag{1.1.3}$$

$$\psi[x(t_f),\ t_f] = 0 \tag{1.1.4}$$

$u(\cdot)$ is a member of the set U, t a member of $[t_o, t_f]$. (1.1.5)

Here x is an n-dimensional state vector and u is an m-dimensional control vector. The functions L and F are scalar and the terminal constraint function ψ is an s-dimensional column vector function of $x(t_f)$ at t_f. The functions L, F and ψ are assumed smooth. The set is defined by

$$U \equiv \{u(\cdot) : u_i(\cdot) \text{ is piecewise continuous in time,}$$

$$|u_i(t)| < \infty, \qquad t_o \leq t \leq t_f,$$

$$i = 1, 2, \ldots, m\}. \tag{1.1.6}$$

The initial time t_o is given explicitly but the final time t_f may be unspecified.

The Hamiltonian for this problem is

$$H(x,u,\lambda,t) = L(x,u,t) + \lambda^T f(x,u,t) \tag{1.1.7}$$

and the following necessary conditions (Pontryagin's principle) hold along an optimal trajectory:

$$-\dot{\lambda} = H_x(\bar{x},\bar{u},\lambda,t) \tag{1.1.8}$$

$$\lambda(t_f) = F_x[\bar{x}(t_f), t_f] + \psi_x^T \nu \qquad (1.1.9)$$

$$H(t_f) = - F_t[\bar{x}(t_f), t_f] - \psi_t^T \nu \qquad (1.1.10)$$

where

$$\bar{u} = \arg\min_u H(\bar{x}, u, \lambda, t) \qquad (1.1.11)$$

$u(\cdot)$ a member of U.

Here $\bar{x}(\cdot)$, $\bar{u}(\cdot)$ denote the candidate state and control functions respectively, $\lambda(\cdot)$ denotes an n-dimensional vector of Lagrange multiplier functions of time, and ν is an s-dimensional vector of Lagrange multipliers associated with ψ.

A singular minimizing arc for the problem of Bolza is defined by Bliss (1946) as one for which the Legendre-Clebsch necessary condition is not satisfied with strict inequality. For the optimal control problem as formulated above the definition of Bliss is equivalent to the following. An extremal arc of the control problem is said to be singular if the $m \times m$ determinant $\det(H_{uu})$ vanishes at any point along it. Otherwise it is said to be nonsingular. In particular, if the Hamiltonian H is linear in one or more elements of the control function then the extremal is singular (Goh, 1966b).

The following definitions are used in the sequel:

Definition 1.1 Let u_k be an optimal singular element of the control vector u on the interval $[t_1, t_2]$ which appears linearly in the Hamiltonian. Let the 2q th time derivative of H_{u_k} be the lowest order total derivative in which u_k appears explicitly with a coefficient which is not identically zero on $[t_1, t_2]$. Then the integer q is called the order of the singular arc. The control variable u_k is referred to as a singular control.

Definition 1.2 Assuming all the elements $u_1, u_2, \ldots, u_m$ of the control vector u are singular simultaneously then u is called a totally singular control function when

$$H_u(\bar{x}, \lambda, t) = 0 \qquad (1.1.12)$$

for all t in $[t_o, t_f]$.

Definition 1.3 A partially singular control function is one along which (1.1.12) holds for k subintervals of length T_i, i = 1, 2, ... , k and where

$\sum_{i=1}^{k} T_i < t_f - t_o$ (Jacobson, 1970b).

The concepts of total and partial singularity can be applied also to the accessory minimum problem in which the second variation of the cost functional (1.1.1) is to be minimized (see Sections 4.4 and 6.1).

In the important subclass of optimal control problems known as relaxed variational problems discussed by Steinberg (1971) some of the elements of the control vector u appear linearly and others appear nonlinearly.

In this book we shall be mainly concerned with problems which are totally singular for all t in $[t_o, t_f]$ (but see Section 5.4). In the partially singular case the nonsingular controls that are present can be eliminated via well-developed non-singular theory, leaving a problem totally singular in the remaining control variables (Robbins, 1967). The singular control theory to be discussed has been developed for and motivated by engineering problems. Nevertheless, singular problems may arise in any discipline where control theory is applied, evidence for which can be seen in economics (Dobell and Ho, 1967) and production from natural resources (Goh, 1969/70). In the following chapters necessary and sufficient conditions will be developed for singular optimal control of systems governed by ordinary differential equations. However, investigation is being carried out into singular control of discrete time systems (Tarn et al., 1971; Graham and D'Souza, 1970) and of systems with delay (Soliman and Ray, 1972; Connor, 1974).

1.2 Singular Control in Space Navigation

Practical problems involving singular controls arose early in the study of optimal trajectories for space manoeuvres. Trajectories for rocket propelled vehicles in which the thrust magnitude is bounded exhibit singularity in the rate of fuel consumption. Lawden (1963) formulates the fundamental problem of space navigation in the following way. $Ox_1x_2x_3$ is an inertial frame in which a space vehicle has position coordinates (x_1, x_2, x_3) and velocity components (v_1, v_2, v_3) at time t. The rocket thrust has direction cosines (ℓ_1, ℓ_2, ℓ_3) and the gravitational field has components (g_1, g_2, g_3). The mass rate of propellant consumption, bounded above by some finite constant $\bar{m}$, is denoted by m. If the rocket has an exhaust velocity c and the mass of the vehicle is M then its equations of motion are

$$\dot{v}_i = cm\ell_i/M + g_i(x_1, x_2, x_3, t) \qquad (1.2.1)$$

$$\dot{x}_i = v_i, \qquad i = 1, 2, 3 \qquad (1.2.2)$$

$$\dot{M} = - m. \qquad (1.2.3)$$

The propellant consumption rate m must satisfy the inequality constraints

$$0 \leq m \leq \bar{m} \qquad (1.2.4)$$

whilst the direction cosines may be written in the form

$$\ell_1 = \sin\theta \cos\phi, \quad \ell_2 = \sin\theta \sin\phi, \ell_3 = \cos\theta \qquad (1.2.5)$$

where θ, ϕ are spherical polar coordinates. It is required to choose the control variables $m(\cdot)$, $\theta(\cdot)$, $\phi(\cdot)$ in order to minimize the fuel used in transferring the vehicle between two given positions at each of which the vehicle's velocity is specified. The final time t_f may or may not be given explicitly. State variables $x_i(\cdot)$, $v_i(\cdot)$, $M(\cdot)$ must satisfy boundary conditions

$$x_i(t_o) = x_{io}, \quad v_i(t_o) = v_{io}, \quad M(t_o) = M_o \qquad (1.2.6)$$

$$x_i(t_f) = x_{if}, \quad v_i(t_f) = v_{if} \qquad (1.2.7)$$

and the cost functional can be written as

$$\mathcal{J} = - M(t_f). \qquad (1.2.8)$$

In this problem the Hamiltonian is linear in the rate of fuel consumption and this control variable turns out to be a singular control.

During the 1950's it was not known whether a given manoeuvre in space using a small, continuous thrust would be more economical in fuel than the previously accepted optimal procedure of impulsive boosts (Hohmann transfers). Some numerical results (Forbes, 1950) suggested that in certain circumstances less fuel would be used in an orbital transfer by following a spiral path using a small but continuous thrust throughout the maneouvre than by a Hohmann transfer. Other analysis (Lawden, 1950, 1952) appeared to show that the so-called intermediate-thrust arcs (along which the fuel expenditure rate is non-zero but less than $\bar{m}$) were inadmissible in a fuel optimal trajectory. The true status of the intermediate-thrust arcs remained hidden until the mathematical theory had been further developed (see Section 1.5.1 and Chapter 3). Much of the early work in aerospace has been surveyed elsewhere (Bell, 1968).

1.3 Method of Miele via Green's Theorem

A method which deals successfully with problems involving singular controls is due to Miele (1950-51). It is based upon a transformation using Green's theorem relating line and surface integrals. As developed by Miele the method is applicable only to a particular class of linear problems in two dimensions. The problems are linear in the sense that the cost functional and any isoperimetric constraint are linear in the derivative of the dependent variable. The general cost functional in this class of problems can be written as