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调和分析导论

(英文版 · 第3版)

AN INTRODUCTION TO HARMONIC ANALYSIS

THIRD EDITION

Yitzhak Katznelson

(美) Yitzhak Katznelson 著
斯坦福大学

机械工业出版社
China Machine Press

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Preface

Harmonic analysis is the study of objects (functions, measures, etc.), defined on topological groups. The group structure enters into the study by allowing the consideration of the translates of the object under study, that is, by placing the object in a translation-invariant space. The study consists of two steps. First: finding the "elementary components" of the object, that is, objects of the same or similar class, which exhibit the simplest behavior under translation and which "belong" to the object under study (harmonic or spectral *analysis*); and second: finding a way in which the object can be construed as a combination of its elementary components (harmonic or spectral *synthesis*).

The vagueness of this description is due not only to the limitation of the author but also to the vastness of its scope. In trying to make it clearer, one can proceed in various ways*; we have chosen here to sacrifice generality for the sake of concreteness. We start with the circle group $\mathbb{T}$ and deal with classical Fourier series in the first five chapters, turning then to the real line in Chapter VI and coming to locally compact abelian groups, only for a brief sketch, in Chapter VII. The philosophy behind the choice of this approach is that it makes it easier for students to grasp the main ideas and gives them a large class of concrete examples which are essential for the proper understanding of the theory in the general context of topological groups. The presentation of Fourier series and integrals differs from that in [1], [7], [8], and [28] in being, I believe, more explicitly aimed at the general (locally compact abelian) case.

The last chapter is an introduction to the theory of commutative Banach algebras. It is biased, studying Banach algebras mainly as a tool in harmonic analysis.

This book is an expanded version of a set of lecture notes written

*Hence the indefinite article in the title of the book.

for a course which I taught at Stanford University during the spring and summer quarters of 1965. The course was intended for graduate students who had already had two quarters of the basic "real-variable" course. The book is on the same level: the reader is assumed to be familiar with the basic notions and facts of Lebesgue integration, the most elementary facts concerning Borel measures, some basic facts about holomorphic functions of one complex variable, and some elements of functional analysis, namely: the notions of a Banach space, continuous linear functionals, and the three key theorems—"the closed graph", the Hahn-Banach, and the "uniform boundedhess" theorems. All the prerequisites can be found in [23] and (except, for the complex variable) in [22]. Assuming these prerequisites, the book, or most of it, can be covered in a one-year course. A slower moving course or one shorter than a year may exclude some of the starred sections (or subsections). Aiming for a one-year course forced the omission not only of the more general setup (non-abelian groups are not even mentioned), but also of many concrete topics such as Fourier analysis on $\mathbb{R}^n, n > l$, and finer problems of harmonic analysis in $\mathbb{T}$ or $\mathbb{R}$ (some of which can be found in [13]). Also, some important material was cut into exercises, and we urge the reader to do as many of them as he can.

The bibliography consists mainly of books, and it is through the bibliographies included in these books that the reader is to become familiar with the many research papers written on harmonic analysis. Only some, more recent, papers are included in our bibliography. In general we credit authors only seldom—most often for identification purposes. With the growing mobility of mathematicians, and the happy amount of oral communication, many results develop within the mathematical folklore and when they find their way into print it is not always easy to determine who deserves the credit. When I was writing Chapter III of this book, I was very pleased to produce the simple elegant proof of Theorem III.1.6 there. I could swear I did it myself until I remembered two days later that six months earlier, "over a cup of coffee," Lennart Carleson indicated to me this same proof.

The book is divided into chapters, sections, and subsections. The chapter numbers are denoted by roman numerals and the sections and subsections, as well as the exercises, by arabic numerals. In cross references within the same chapter, the chapter number is omitted; thus Theorem III.1.6, which is the theorem in subsection 6 of Section 1 of Chapter III, is referred to as Theorem 1.6 within Chapter III, and

Theorem III.1.6 elsewhere. The exercises are gathered at the end of the sections, and exercise V.1.1 is the first exercise at the end of Section 1, Chapter V. Again, the chapter number is omitted when an exercise is referred to within the same chapter. The ends of proofs are marked by a triangle (◀).

The book was written while I was visiting the University of Paris and Stanford University and it owes its existence to the moral and technical help I was so generously given in both places. During the writing I have benefited from the advice and criticism of many friends; I would like to thank them all here. Particular thanks are due to L. Carleson, K. DeLeeuw, J.-P. Kahane, O.C. McGehee, and W. Rudin. I would also like to thank the publisher for the friendly cooperation in the production of this book.

YITZHAK KATZNELSON

Jerusalem
April 1968

The third edition

The second edition was essentially identical with the first, except for the correction of a few misprints. In the current edition some more misprints were corrected, the wording changed in a few places, and some material added: two additional sections in Chapter I and one in Chapter IV; an additional appendix; and a few additional exercises.

The added material does not reflect the progress in the field in the past thirty or forty years. Much of it could and, in retrospect, should have been included in the first edition of the book.

This book was and is intended to serve, as its title makes explicit, as an introduction. It offers what I believe to be the core material and technique, a basis on which much can be built.

The added items in the bibliography expand on parts which are discussed here only briefly (or not at all), and provide a much more up-to-date bibliography of Harmonic analysis.

Y. K.

Stanford
June 2003

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$BV(\mathbb{T})$, 17
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Chapter I

Fourier Series on $\mathbb{T}$

We denote by $\mathbb{R}$ the additive group of real numbers and by $\mathbb{Z}$ the subgroup consisting of the integers. The group $\mathbb{T}$ is defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$ where, as indicated by the notation, $2\pi\mathbb{Z}$ is the group of the integral multiples of 2π. There is an obvious identification between functions on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$, which allows an implicit introduction of notions such as continuity, differentiability, etc., for functions on $\mathbb{T}$. The *Lebesgue measure* on $\mathbb{T}$ can be equally defined by means of the preceding identification: a function f is integrable on $\mathbb{T}$ if the corresponding 2π-periodic function, which we denote again by f, is integrable on $[0, 2\pi)$ and we set

$$\int_{\mathbb{T}} f(t)dt = \int_0^{2\pi} f(x)dx.$$

In other words, we consider the interval $[0, 2\pi)$ as a model for $\mathbb{T}$ and the Lebesgue measure dt on $\mathbb{T}$ is the restriction of the Lebesgue measure of $\mathbb{R}$ to $[0, 2\pi)$. The total mass of dt on $\mathbb{T}$ is equal to 2π and many of our formulas would be simpler if we normalized dt to have total mass 1, that is, if we replace it by $dx/2\pi$. Taking intervals on $\mathbb{R}$ as "models" for $\mathbb{T}$ is very convenient, however, and we choose to put $dt = dx$ in order to avoid confusion. We "pay" by having to write the factor $1/2\pi$ in front of every integral.

An all-important property of dt on $\mathbb{T}$ is its *translation invariance,* that is, for all $t_0 \in \mathbb{T}$ and f defined on $\mathbb{T}$,

$$\int f(t - t_0)dt = \int f(t)dt^{\dagger}.$$

†Throughout this chapter, integrals with unspecified limits of integration are taken over $\mathbb{T}$.

1 FOURIER COEFFICIENTS

1.1 We denote by $L^1(\mathbb{T})$ the space of all (equivalence† classes of) complex-valued, Lebesgue integrable functions on $\mathbb{T}$. For $f \in L^1(\mathbb{T})$ we put

$$\|f\|_{L^1} = \frac{1}{2\pi}\int_{\mathbb{T}} |f(t)|dt.$$

It is well known that $L^1(\mathbb{T})$, with the norm so defined, is a Banach space.

DEFINITION: A *trigonometric polynomial* on $\mathbb{T}$ is an expression of the form

$$P \sim \sum_{n=-N}^{N} a_n e^{int}. \tag{1.1}$$

The numbers n appearing in (1.1) are called the frequencies of P; the largest integer n such that $|a_n|+|a_{-n}| \neq 0$ is called *the degree of* P. The values assumed by the index n are integers so that each of the summands in (1.1) is a function on $\mathbb{T}$. Since (1.1) is a finite sum, it represents a function, which we denote again by P, defined for each $t \in \mathbb{T}$ by

$$P(t) = \sum_{n=-N}^{N} a_n e^{int}. \tag{1.2}$$

Let P be defined by (1.2). Knowing the function P we can compute the coefficients a_n by the formula

$$a_n = \frac{1}{2\pi}\int P(t)e^{-int}dt \tag{1.3}$$

which follows immediately from the fact that for integers j,

$$\frac{1}{2\pi}\int e^{ijt}dt = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases}$$

Thus we see that the function P determines the expression (1.1) and there seems to be no point in keeping the distinction between the expression (1.1) and the function P; we shall consider trigonometric polynomials as both formal expressions and functions.

† $f \sim g$ if $f(t) = g(t)$ almost everywhere.

1.2 DEFINITION: A *trigonometric series* on $\mathbb{T}$ is an expression of the form

$$S \sim \sum_{n=-\infty}^{\infty} a_n e^{int}. \tag{1.4}$$

Again, n assumes integral values; however, the number of terms in (1.4) may be infinite and there is no assumption whatsoever about the size of the coefficients or about convergence. The conjugate† of the series (1.4) is, by definition, the series

$$\tilde{S} \sim \sum_{n=-\infty}^{\infty} -i \operatorname{sgn}(n) a_n e^{int},$$

where $\operatorname{sgn}(n) = 0$ if $n = 0$ and $\operatorname{sgn}(n) = n/|n|$ otherwise.

1.3 Let $f \in L^1(\mathbb{T})$. Motivated by (1.3) we define the nth Fourier coefficient of f by

$$\hat{f}(n) = \frac{1}{2\pi} \int f(t) e^{-int} dt. \tag{1.5}$$

DEFINITION: *The Fourier series $S[f]$ of a function $f \in L^1(\mathbb{T})$* is the trigonometric series

$$S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}.$$

The series conjugate to $S[f]$ will be denoted by $\widetilde{S}[f]$ and referred to as the *conjugate Fourier series* of f. We shall say that a trigonometric series is a Fourier series if it is the Fourier series of some $f \in L^1(\mathbb{T})$.

1.4 We turn to some elementary properties of Fourier coefficients.

Theorem. *Let $f, g \in L^1(\mathbb{T})$, then*

(a) $\widehat{(f+g)}(n) = \hat{f}(n) + \hat{g}(n)$.

(b) For any complex number α

$$\widehat{(\alpha f)}(n) = \alpha \hat{f}(n).$$

† See Chapter III for motivation of the terminology.

(c) If $\bar{f}$ *is the complex conjugate*[‡] *of* f *then* $\hat{\bar{f}}(n) = \overline{\hat{f}(-n)}$.

(d) Denote $f_\tau(t) = f(t-\tau)$, $\tau \in \mathbb{T}$; *then*

$$\hat{f}_\tau(n) = \hat{f}(n)e^{-in\tau}.$$

(e) $|\hat{f}(n)| \le \frac{1}{2\pi}\int |f(t)|dt = \|f\|_{L^1}$.

The proofs of (a) through (e) follow immediately from (1.5) and the details are left to the reader.

1.5 *Corollary.* *If* $f_j \in L^1(\mathbb{T})$, $j = 0, 1, \ldots$, *and* $\|f_j - f_0\|_{L^1} \to 0$, *then* $\hat{f}_j(n) \to \hat{f}_0(n)$ *uniformly.*

1.6 *Theorem.* *Let* $f \in L^1(\mathbb{T})$, *assume* $\hat{f}(0) = 0$, *and define*

$$F(t) = \int_0^t f(\tau)d\tau.$$

Then F *is continuous,* 2π*-periodic, and*

$$\hat{F}(n) = \frac{1}{in}\hat{f}(n), \quad n \neq 0. \tag{1.6}$$

PROOF: The continuity (and, in fact, the absolute continuity) of F is evident. The periodicity follows from

$$F(t+2\pi) - F(t) = \int_t^{t+2\pi} f(\tau)d\tau = 2\pi\hat{f}(0) = 0,$$

and (1.6) is obtained through integration by parts:

$$\hat{F}(n) = \frac{1}{2\pi}\int_0^{2\pi} F(t)e^{-int}dt = \frac{-1}{2\pi}\int_0^{2\pi} F'(t)\frac{1}{-in}e^{-int}dt = \frac{1}{in}\hat{f}(n).$$

◀

1.7 We now define the convolution operation in $L^1(\mathbb{T})$. The reader will notice the use of the group structure of $\mathbb{T}$ and of the invariance of dt in the subsequent proofs.

[‡]Defined by: $\bar{f}(t) = \overline{f(t)}$ for all $t \in \mathbb{T}$.

Theorem. *Let $f, g \in L^1(\mathbb{T})$. For almost all t, the function $f(t-\tau)g(\tau)$ is integrable (as a function of τ on $\mathbb{T}$) and, if we write*

$$h(t) = \frac{1}{2\pi}\int f(t-\tau)g(\tau)d\tau, \tag{1.7}$$

then $h \in L^1(\mathbb{T})$ and

$$\|h\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}. \tag{1.8}$$

Moreover

$$\hat{h}(n) = \hat{f}(n)\hat{g}(n) \quad \textit{for all } n. \tag{1.9}$$

PROOF: The functions $f(t-\tau)$ and $g(\tau)$, considered as functions of the two variables (t, τ), are clearly measurable, hence so is

$$F(t,\tau) = f(t-\tau)g(\tau).$$

For every τ, $F(t,\tau)$ is just a constant multiple of f_τ, hence integrable dt, and

$$\frac{1}{2\pi}\int\left(\frac{1}{2\pi}\int|F(t,\tau)|dt\right)d\tau = \frac{1}{2\pi}\int|g(\tau)|\;\|f\|_{L^1}d\tau = \|f\|_{L^1}\|g\|_{L^1}.$$

Hence, by the theorem of Fubini, $f(t-\tau)g(\tau)$ is integrable (over $(0, 2\pi)$) as a function of τ for almost all t, and

$$\begin{aligned}\frac{1}{2\pi}\int|h(t)|dt = \frac{1}{2\pi}\int\left|\frac{1}{2\pi}\int F(t,\tau)d\tau\right|dt &\le \frac{1}{4\pi^2}\iint|F(t,\tau)|dt\,d\tau \\ &= \|f\|_{L^1}\|g\|_{L^1},\end{aligned}$$

which establishes (1.8). In order to prove (1.9) we write

$$\begin{aligned}\hat{h}(n) &= \frac{1}{2\pi}\int h(t)e^{-int}dt = \frac{1}{4\pi^2}\iint f(t-\tau)e^{-in(t-\tau)}g(\tau)e^{-in\tau}dt\,d\tau \\ &= \frac{1}{2\pi}\int f(t)e^{-int}dt\cdot\frac{1}{2\pi}\int g(\tau)e^{-in\tau}d\tau = \hat{f}(n)\hat{g}(n).\end{aligned}$$

As above, the change in the order of integration is justified by Fubini's theorem. ◀

1.8 DEFINITION: The *convolution* $f * g$ of the $(L^1(\mathbb{T}))$ functions f and g is the function h defined by (1.7). Using the star notation for the convolution, we can write (1.9) as

$$\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n). \tag{1.10}$$

Theorem. *The convolution operation in $L^1(\mathbb{T})$ is commutative, associative, and distributive (with respect to the addition).*

PROOF: The change of variable $\vartheta = t - \tau$ gives

$$\frac{1}{2\pi}\int f(t-\tau)g(\tau)d\tau = \frac{1}{2\pi}\int g(t-\vartheta)f(\vartheta)d\vartheta,$$

that is,

$$f * g = g * f.$$

If $f_1, f_2, f_3 \in L^1(\mathbb{T})$, then

$$[(f_1 * f_2)*f_3](t) = \frac{1}{4\pi^2}\iint f_1(t-u-\tau)f_2(u)f_3(\tau)du\,d\tau$$
$$=\frac{1}{4\pi^2}\iint f_1(t-\omega)f_2(\omega-\tau)f_3(\tau)d\omega\,d\tau = [f_1 * (f_2 * f_3)](t).$$

Finally, the distributive law

$$f_1 * (f_2 + f_3) = f_1 * f_2 + f_1 * f_3$$

is evident from (1.7). ◀

1.9 ***Lemma.*** *Assume $f \in L^1(\mathbb{T})$ and let $\varphi(t) = e^{int}$ for some integer n. Then*

$$(\varphi * f)(t) = \hat{f}(n)e^{int}.$$

PROOF:

$$(\varphi * f)(t) = \frac{1}{2\pi}\int e^{in(t-\tau)}f(\tau)d\tau = e^{int}\frac{1}{2\pi}\int f(\tau)e^{-in\tau}d\tau.$$

◀

Corollary. *If $f \in L^1(\mathbb{T})$ and $k(t) = \sum_{-N}^{N} a_n e^{int}$, then*

$$(k * f)(t) = \sum_{-N}^{N} a_n\hat{f}(n)e^{int}. \tag{1.11}$$

EXERCISES FOR SECTION 1

1.1. Compute the Fourier coefficients of the following functions (defined by their values on $[-\pi, \pi)$):

(a)
$$f(t) = \begin{cases} \sqrt{2\pi} & |t| < \frac{1}{2} \\ 0 & \frac{1}{2} \le |t| \le \pi. \end{cases}$$

(b)
$$\Delta(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & 1 \le |t| \le \pi. \end{cases}$$

What relation do you see between f and Δ ?

(c)
$$g(t) = \begin{cases} 1 & -1 < t \le 0 \\ -1 & 0 < t < 1 \\ 0 & 1 \le |t|. \end{cases}$$

What relation do you see between g and Δ?

(d)
$$h(t) = t \qquad -\pi < t < \pi.$$

1.2. Remembering Euler's formulas

$$\cos t = \frac{1}{2}(e^{it} + e^{-it}), \qquad \sin t = \frac{1}{2i}(e^{it} - e^{-it}),$$

or

$$e^{it} = \cos t + i \sin t\,,$$

show that the Fourier series of a function $f \in L^1(\mathbb{T})$ is formally equal to

$$\frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos nt + B_n \sin nt)$$

where $A_n = \hat{f}(n) + \hat{f}(-n)$ and $B_n = i(\hat{f}(n) - \hat{f}(-n))$. Equivalently:

$$A_n = \frac{1}{\pi}\int f(t) \cos nt\, dt$$
$$B_n = \frac{1}{\pi}\int f(t) \sin nt\, dt.$$

Show also that if f is real valued, then A_n and B_n are all real; if f is even, that is, if $f(t) = f(-t)$, then $B_n = 0$ for all n; and if f is odd, that is, if $f(t) = -f(-t)$, then $A_n = 0$ for all n.

1.3. Show that if $S \sim \sum a_j \cos jt$, then $\tilde{S} \sim \sum a_j \sin jt$.