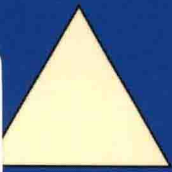

Beautiful

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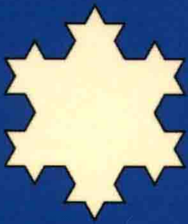

Exact

Crazy

Mathematics in the Real World

Apoorva Khare and Anna Lachowska

Beautiful, Simple, Exact, Crazy: Mathematics in the Real World

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Beautiful, Simple, Exact, Crazy

Preface

The idea for this book arose out of an introductory mathematics course, "Mathematics in the Real World", that the authors co-designed and have taught at Yale and Stanford. The purpose of the course is to familiarize students whose primary interests lie outside of the sciences with the power and beauty of mathematics. In particular, we hope to show how simple mathematical ideas can be applied to answer real-world questions.

Thus we see this as a college-level course book that can be used to teach basic mathematics to students with varying skill levels. We discuss specific and relevant real-life examples: population growth models, logarithmic scales, personal finance, motion with constant speed or constant acceleration, computer security, elements of probability, and statistics. Our goal is to combine the right level of difficulty, pace of exposition, and scope of applications for a curious liberal arts college student to study and enjoy.

Additionally, the book could find use by high school students and by anyone wishing to study independently. The prerequisite is only a high school course in algebra. We hope our book can help readers without extensive mathematical training to analyze datasets and real-world phenomena, and to distinguish statements that are mathematically reasonable from those only pretending to be.

Philosophy and goals

Imagine the following dialog in a high school algebra class.

Teacher: "Find the sum of the geometric series $1 + 5 + 5^2 + \ldots + 5^{20}$."

Student (looking out the window and thinking that life is so much bigger than math, and wondering why the class has to suffer through this tedious, long, and pointless computation): "Can't we just type it into a calculator?"

Whatever the teacher's answer, the student knows that he is right at least on one count – life is indeed bigger than math. It is also bigger than physics, history, and sociology. But for some reason the irrelevance of mathematics is much easier to accept. It is not acceptable to say, "I do not care who was the first president of the United States," but it seems just fine to say, "I do not care about the purpose of a geometric series." It is not acceptable to brag, "I'm illiterate!" But many people feel justified telling friends that they cannot understand mathematics. Yet the consequences of refusing to learn even the most basic mathematical ideas are dire both for the individual and for society.[1] As a society, we need to make collective decisions based on information provided by the media and other sources of variable reliability – and the quality of such decisions depends on our understanding of logic and statistics. As individuals, we have to cope with personal finance and Internet security. We have to know how to estimate the chances an event will occur, and how to interpret lots of new information. From a less practical viewpoint, mathematics adds another dimension (or two, or twenty-three[2]) to the way we see the world, which might be a source of inspiration for a person of any occupation.

The student we just encountered likely will come to college to major in the humanities or social sciences, and is part of the target audience for this book. Thus, our main goal is to convince our reader that mathematics can be easy, its applications are real and widespread, and it can be amusing and inspiring.

To illustrate, let us return to our geometric series example. Mathematics is known to have formulas for everything, so it is not surprising that it has a formula for the sum $1 + 5 + 5^2 + \ldots + 5^n$ for any natural n. What might be more surprising is that this formula requires no derivatives, integrals, or trigonometric identities – and the computation (given in Chapter 9) takes only one line.

Practical applications of this simple mathematical concept are everywhere: for instance, it governs the payment of mortgages and car loans for millions of people. There is more! Everyone knows that $\frac{1}{3} = 0.333333...$, but the fact that *any* repeating decimal, say, $0.765765765...$, can be easily written as a fraction of two integers follows from the formula for the geometric series. The same formula lies in the foundation of Zeno's paradox of Achilles and a tortoise, covered in Chapter 10, and two centuries later, another Greek, Archimedes, used a geometric series to calculate the area

[1] We will not elaborate on this here, referring the reader, for example, to *Innumeracy* by J.A. Paulos, published by Hill and Wang, New York, 2001.

[2] According to string theory, the world might have 26 dimensions.

under a parabolic arc. Why should we care? If understanding history is a good enough reason, we can recall that by some accounts, Archimedes also took part in the defense of his town of Syracuse during the siege by the Romans (214-212 BC), and might have used his computation to design a parabolic mirror to set the enemy's fleet on fire. Now, fast-forward to the twentieth century. "How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension" was the title of the article, published in *Science* in 1967 by the French-American mathematician Benoit Mandelbrot, that opened a new area of study – fractal geometry. The answer to the question in the title depends on how closely we look, and the closer we look, the greater the coast's length becomes. The same is true for many fractal curves that are easy to describe in words but impossible to draw *precisely*. One of the central tools used to understand these objects is the geometric series.

This is just a glimpse of the scope of applications of one simple mathematical idea – from personal finance, to philosophical puzzles, to fractals, objects of such breathtaking beauty that they make the boundary between art and science disappear.

This example encapsulates our philosophy for the book: we would like to show that there is a lot of simple mathematics relevant to people's everyday lives and their creative aspirations.

There is a way to teach a future artist, and there is a way to teach a layperson to appreciate art. In the latter case, the student is not required to be able to draw and paint like a master, but only to see the beauty in the works of others. Our intention for this book is more ambitious: we hope that it can serve as a guide to the world of mathematics, and also as an inspiration for readers to try their hand at developing and solving mathematical models for their own needs. From there it is only one more step to seeing the world from a mathematical point of view.

The book includes many examples and practice problems designed to gradually build students' proficiency and encourage their involvement with the material. The ultimate goal is to make even the students who are "numerically shy" at the start of a course comfortable applying their mathematical skills in a wide range of situations, from solving puzzles to analyzing statistical data.

We also hope that they become "math-friendly," admitting that mathematics can be interesting and cool. We feel that such comfort levels are indeed achieved when we teach this course at Yale and Stanford – based on our discussions with students, their exams and homework, and their feedback from the course.

Contents and structure

Here are the most important features that distinguish our approach.

First, we neither discuss nor assume any knowledge of calculus or trigonometry. There are quite a few simple concepts in mathematics that do not depend on this particular knowledge but have enormous importance in the real world, as a vast number of pertinent applications show.

Second, our preference is for a self-contained linear exposition, devoid of digressions and asides. We also refrain from presenting an overload of pictures, data analysis, or complicated examples. This book should be viewed as supporting material for a first course in college-level mathematics; there is ample opportunity to analyze more complex examples and phenomena in future studies.

The choice of topics in the book was governed by the following principles:

1. The mathematics involved should be simple, accessible to a student with no experience beyond elementary high school algebra, and explainable within a couple of pages.

2. These simple mathematical concepts should generate a wide range of practical, impressive, or amusing real-world applications.

Accordingly, each chapter of the book, starting with Chapter 2, is divided into two parts:

1. The first, shorter part contains the necessary mathematics: definitions, statements, explanations, examples. This part is supposed to be *studied*, or read slowly, with the reader occasionally doing suggested computations. It can also play the role of lecture notes, if the book is used in teaching a course. Finally, the clearly distinct math section can be used as a reference when reading about applications.

2. The second part, which constitutes most of the chapter, is an exploration of various real-world applications. It contains questions posed and answered by means of the mathematical tools presented in the first part. The second part has little or no mathematical argument and is designed to be read leisurely.

The math sections are there for studying the necessary mathematics. The applications sections are for reading about ways this math can be used. By separating these two activities, we hope to promote learning; instead of students half paying attention while reading fifty pages of a typical chapter of a basic math textbook, we require intense concentration on just a few pages, and then they can enjoy reading the remaining ten or fifteen pages of applications. We hope that having short math and long applications sections will impress on the students that even when the amount of mathematics to be learned is limited, the potential rewards for this effort can be large.

The structure allows readers to choose their own way through the book: some might skip the math sections entirely, and look only for the applications; they will not learn the mathematics but they might learn what it is good for. Others might like the math sections but choose to read only about some of the suggested applications. In some sense, the book is a hopscotch game where the reader sets the rules.

Our strategy for the choice of topics resulted in an array of subjects, presented in chapters that are only loosely interrelated. This means that even though the structure of the book is linear, a reader (or a teacher) can skip some parts of the book or change their order. The chapter dependencies are:

$$(1), \quad (2-3-4), \quad (5-6-7-8), \quad (9-10),$$

$$(11), \quad (12-13), \quad (14-15-16), \quad (17-18).$$

This makes for greater flexibility in teaching the course. For example, one author taught chapters 1-3, 5, 6, 8, 9-10, 12, 14, 16, 17-18, while the other author taught chapters 2-10, 12-16. Moreover, instructors can introduce their own topics and design additional material based on the conceptual structure presented in the book.

The book is intended, in particular, for "artists and poets," so we include several references to works of literature and art. Chapter 2 contains a mathematical analysis of a short story by an Italian writer, D. Buzzati, which is based on the repeated application of a constant speed motion model. A mathematical perspective on the development of music scales is given in Chapter 7. The same mathematical concept, a logarithm, can be used to illustrate our egocentricity: a famous *New Yorker* cover by S. Steinberg, *View of the World from 9th Avenue*, can be interpreted as a logarithmic scale of distances. In connection with the radiocarbon dating method (Chapter 8), we discuss some of the mysteries of the famous Voynich manuscript, an undeciphered early fifteenth-century text whose author, language, and subject are unknown.

The relation between nature and mathematics is another source of philosophical and poetic inspiration. Logarithmic scales allow us to grasp, at least to some extent, the soul-chilling range of magnitude of distances in the universe. Chapter 10 contains a discussion of the possibility of encountering infinity in reality. And in Chapter 4, the growth patterns in plants are shown to be related to mathematical properties of a certain irrational number, the golden ratio, whose cultural significance is also remarkable.

Perhaps the most important part of the book consists of the wide variety of examples, practice problems, and exercises of varying levels of difficulty. We tried to make sure that all of the mathematics in this book is reinforced by studying concrete examples. Here are the main highlights of the problem-solving component of the book:

- Each chapter contains a number of concrete examples. Many of them are discussed in detail and solved in the text (**Examples**).

- Others are grouped in clusters according to topics within a chapter, and provided with answers (**Practice problems**). These are usually easy questions designed to check students' basic understanding of the concepts presented in the chapter.

- Still others are confined to the end of a chapter and are supposed to be solved independently (**Exercises**). Answers and solutions to the odd-numbered exercises are given at the end of the book.

- In addition we provide sample midterms and final exams and their solutions. For those engaged in independent study, these can be used for self-evaluation and review.

Some sections of the text, examples, and exercises are marked with an asterisk. This indicates that the material discussed in this part of the book is more challenging or abstract, and can be safely skipped at first reading.

Target audience

Our primary goal is to provide a book for an introductory college-level course in mathematics and its applications. Thus, we hope to reach undergraduates looking to balance their humanities and social sciences education with a touch of math. Many colleges have an academic requirement for their students to diversify their choice of classes and include at least a few "hard-science" or "quantitative reasoning" courses. A course based on this book can be one of them. We hope that while helping students to complete their academic requirements, the book will also significantly benefit them in their future careers.

When the course "Mathematics in the Real World" was introduced at Yale University, it received twice as many applications as there was room for, and the course remains quite popular. In course evaluations and thank-you notes, students mention that the course has helped them acquire basic mathematical literacy and confidence, and instead of being scary, it was an enjoyable experience. We hope that this book will elicit similar reactions from some of our readers.

The course has also been introduced at Stanford University. Our long-term hope is that this book will contribute to the regular curriculum in many colleges and universities.

We also hope that our book will be helpful for those who would like to specialize in natural sciences or economics, but who lack some background in pre-calculus and calculus. For them, the book might serve as a first step on their way to more advanced mathematics.

The book might also be useful for advanced high-school students who are interested in real-world applications of the concepts they learn at school. We estimate that the level of exposition is suitable for most high schools, and the book can be used either by teachers, to supplement the standard program, or by students as extracurricular reading.

In addition, the universal appeal of the topics and the minimal mathematical prerequisites needed to understand this text make for a significantly wider audience. This book should be usable for independent study by busy adults who wish to improve their understanding of math (the book is short!). It might also have some appeal for busy but more experienced math fans, who could leisurely scan it for literary references and unusual applications of math concepts.

Finally, we expect the book to be attractive to international audiences. The authors are of Indian and Russian origin, and we tried to give the book an international flavor. Our examples and cultural references are drawn from all over the world.

In this book we tried to keep a balance between being instructional, being entertaining, and being practical. We hope that this approach will help us promote mathematics as an art, a skill, a language, a way of thinking, a game of puzzles, and in general a worthy activity, to a wide audience of readers.

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We would like to thank our parents and our teachers in high school, college, and graduate school, who taught us to appreciate the depth and beauty of mathematics, and encouraged us, by their own example, to share it with others.

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Contents